AOS 610   GFD I   Prof. Hitchman
Key concepts from lectures since the second quiz

I. Waves

Readings
Holton Chapter 7 (linear perturbation theory)
Gill Chapter 5 95-107; 6 117-137, 7 208-214

1. General

Quasi-periodic, can transfer information far
with only moderate parcel displacements
Restoring force: PGF<->density,
gravity in density stratification
meridional gradient of planetary vorticity
Free waves excited by weak random motions (normal modes)
Forced waves continuously forced at matched scale (most waves)
Internal waves propagate inside of a fluid,
sinusoidal oscillations, $\exp(i\, m_r z)$
External waves are edge waves, max amplitude at
boundary, $\exp(-m_i z)$
Stationary ($c_r=0$) versus travelling
Steady ($c_i=0$) versus transient (amplitude varies in time)
Linear (infinitesimal amplitude) versus finite amplitude
Mean flow acceleration will occur if the flow is transient,
diabatic, or nonlinear. If it is steady, frictionless,
and adiabatic it satisfies the non-acceleration and
non-transport conditions.
Dispersive $c=c(k)$
Sound waves $c = \sqrt{\gamma R T}$
Shallow water waves $c = \sqrt{gH}$
Rossby waves $c = U - \beta/k^2$
A dispersion relation relates time and space scales together
according to the fundamental physics of the restoring force;
allows classification of wave type, inference of behavior
Define phase for a plane wave, wavenumber and frequency
as a function of phase, phase velocity, trace velocity,
group velocity

2. External gravity waves

Shallow water waves ($L \gg H$; $kH \ll 1$) are hydrostatic;
use reduced gravity, convert horizontal pressure gradient
force to horizontal slope of surface,
vertically integrated continuity equation, nondispersive,
tsunamis $H\sim 4$ km, $c\sim 200$ m/s
Laplace 1778 deep water waves ($L \ll H$; $kH \gg 1$)
dispersive $c = \sqrt{g/k}$; waves travel twice as fast as energy
$w(0)=0$ and $w(H)=dh/dt$
$\tanh kh \to 1$ for kh large; $\tanh kh \to kh$ for kh small
two limits on general dispersion relation (shallow and deep water)
Two layers: ratio of amplitudes is the separation constant or
eigenvector; two equivalent depths arise - the barotropic
one which acts in unison, behaving like a single layer of
total depth H and fast phase speed $\sqrt{gH}$, while in the baroclinic

mode the two layers act in antiphase, behaving like a single layer with
equivalent depth He and slow phase speed ce = sqrt(g He)=sqrt(g rho’/rho H1)
Stokes drift; Lagrangian = Eulerian + Stokes drift
 due to amplitude variation in space

3. Internal gravity waves

Internal gravity waves exist in continuous stratification
omega = U k +/- Nk/sqrt(k^2 + m^2)
 Energy travels many km while parcels travel only a few meters
 Tides, lee waves, Kelvin waves are all examples
 N (buoyancy cutoff) > omega > f (inertial cutoff)
 dispersion relation depends on static stability and
  slope of parcel displacements
 horizontal trace speed and group speed are the same
 vertical trace speed is opposite to vertical group velocity,
  very dispersive; energy leaves source, usually upward,
  individual wave crests form on leading edge and propagate
  down through the field, where they disappear off the bottom
  of the ascending energy packet, or envelope
Mountain waves are forced best when the wave is traveling at
 the same speed as the mountain, G_x = c_x = 0 -> U = N/|m|. For Gz > 0
 need m > 0, tilting upstream in westerly flow; High theta,
 high momentum air descends to the surface in a large amplitude
 breaking event. rho u'w' < 0, upward flux of easterly momentum
 or downward flux of westerly momentum occurs when the
 waves break and are absorbed into the mean flow, causing
 the flow to slow down toward the speed of the mountain.

4. Rossby waves

Rossby wave oscillations are perpendicular to their direction
 of propagation (transverse wave) and their flow is perpendicular
 to the the pressure gradient force (geostrophic). These apply
 at large scales (Ro small). The Rossby wave propagation
 mechanism depends on conservation of absolute vorticity. Since
 planetary vorticity, f, varies with latitude, most strongly
 near the equator, this is also known as the beta effect.
A northward-displaced parcel will acquire anticyclonic relative
 vorticity, displacing the parcel to its west northward, and so on.
 A streamline may be viewed as elastic in this sense.
The barotropic dispersion relation is omega = U k – beta k/(k^2 + l^2)
The second term is the Rossby wave propagation relative to the zonal
 mean flow, which is toward the west relative to the mean flow.
 Long waves (small k) can travel upstream
 in midlatitude westerlies fast enough to have the
 wave be stationary (c=0) or retrogressing (c < 0). Short waves
 (large k) travel upstream only slowly, so they are carried
 downstream by the flow and c > 0.
The energy travels downstream faster than the mean flow. This is
 why successive downstream development can be a useful forecasting
 tool. A Hovmoller diagram (x,t) can show slower progression
 of troughs an ridges and faster progression of Rossby wave groups.
Charney-Drazin criterion: As U-c --> 0, l,m --> infinity, as wave
 crests pile up at a critical surface.
 In general, Rossby waves cannot propagate

energy upward into easterlies, the wave energy is absorbed and the wave's momentum is deposited into the mean flow, causing easterly acceleration. As U-c --> Umax = beta/k^2, l,m --> 0. This is a turning point.
Shorter Rossby waves have a smaller Umax, which causes their energy to refract back into the troposphere in strong winter westerlies, leaving planetary scale waves to dominate the stratosphere. In the southern winter even the longest waves are prohibited and one has a deep, strong, nearly symmetric polar vortex.

II. Rossby adjustment problem

Sir Napier Shaw suggested in 1906 that the atmosphere and ocean try to achieve geostrophic balance by radiating gravity waves away.
A small region where f=constant is studied with an initial mass field disturbance. On a rotating sphere a final steady state solution is obtained with a non-zero geostrophic flow.

Shallow water equations. Any solution h will solve the steady state set, making it degenerate (non-unique final state). Laplace solved this set to create a wave equation for Poincare waves (combined N and f in shallow water). Kelvin (1879) created a vorticity equation from this set and showed that eddy potential vorticity is conserved. Rossby (1938) solved this for an initial non-zero distribution of h.

Shallow water PV is conserved. Only some of the APE can be used in the geostrophic adjustment process.
P, Ertel's PV
q, Quasi-geostrophic PV, a linearized version of Ertel's PV.
Conservation of q leads readily to the Rossby wave dispersion relation.

Rossby radius of deformation L_R.
For x < L_R transient gravity waves radiate away from the region, leaving behind a permanent, balanced vortex. For a wave characterized by geostrophic flow (Rossby waves), KE' > PE' for L_x < L_R, while for longer, planetary waves more energy is in the mass distribution (PE').
L_R ~ 500 km in the atmosphere and ~50 km in the ocean.

Prandtl's ratio N/f is linked to the aspect ratio of circulations in the atmosphere or ocean, due to the relative strength of resistance to motion.

Baroclinic life cycle experiments show that IGW radiate upward and away from the leading edge of intrusive air surges in the upper troposphere associated with the synoptic flow. These are absorbed in the lower stratosphere, where they cause vertical displacements and irreversible mixing. These IGW also organize rain bands within the synoptic system. If IGW energy reaches the cold polar night stratosphere, upward motion in one phase of the wave can cause a polarstratospheric cloud, which catalyzes chlorine-induced ozone loss.

III. Baroclinic instability

1. Mechanism

This wave instability works by a growing a perturbation pattern converting available potential energy to eddy energy. This requires a horizontal temperature contrast, or, through the thermal wind relation, vertical shear of the horizontal wind. Because the earth is rotating, this occurs in an organized fashion. A small pressure perturbation can grow because the flow is quasi-geostrophic: poleward rising warm air to the east of the low, equatorward sinking cold air to the west of the low. Since high heights lie over warm air and low heights lie over cold air, the circulation around the growing low pressure center will lead to a westward tilt with height for the system.

The wave can grow if parcel trajectories are sloped less steeply than theta surfaces. Then parcels can continue to accelerate, reducing the center of mass and increasing eddy KE. If the wave is too short horizontally, parcel trajectories will be too steep, up the theta gradient. So static stability will stabilize a baroclinic wave that is too short: the short-wave cutoff.

The waves grow from a horizontal temperature gradient (APE). Due to the thermal wind law, baroclinic waves grow in vertical shear. The result is a poleward heat flux that reduces the equator-pole temperature gradient, thereby reducing the wind shear, so that the tendency for instability is reduced.

2. Eady problem (1949)

Boussinesq, U_z, $w'=0$ at top and bottom, f-plane
(not a Rossby wave in the sense of westward propagation relative to the mean flow due to the beta effect.)
The rigid upper lid behaves somewhat like the tropopause, where upper level PV anomalies in conjunction with a strong surface temperature gradient can rapidly grow strong baroclinic waves in the real world. In the Eady model low and high pressure systems also tilt westward with altitude. Warm air ascends poleward to the east of the low. The Rossby height $H_R=f/(Nk)$ gives the vertical e-folding scale for the influence of a perturbation on the upper or lower lid into the interior. Growth rate is proportional to wind shear. In the atmosphere, the short-wave cutoff is about 2800 km, while the wavelength of fastest growth is about 4000 km, somewhat longer than observed growing baroclinic waves. For typical shears, growth time scales are about 2 days. In the ocean scales are about 200 km, with a 100 day growth time scale. The horizontal scale of maximum growth is about 3 times L_R.

3. Charney problem (1947)

No rigid upper lid, beta plane. Rossby wave dynamics (beta effect) are included and the extension into the stratosphere is more realistic. A long-wave cutoff arises in this problem, due to the beta effect. At sufficiently long horizontal scales, advection of planetary vorticity dominates over advection of relative vorticity. Therefore, west of the trough aloft, equatorward flow advects high planetary vorticity from higher latitudes. This will tend to increase the relative vorticity aloft over the high, which can only be accomplished by horizontal divergence aloft, which will diminish the surface high. To the east of the trough aloft, poleward flow advects low planetary vorticity, which induces horizontal convergence, weakening the surface low. The result is to weaken the surface H and L, hence eddy winds, and ability to convert APE to eddy KE.

In the real atmosphere curved flows with inflection points in the vertical and horizontal lead to combined barotropic/baroclinic growth or decay, depending on the phase relationship with its environment. Latent heating in the warm upglide sector can enhance this motion, while infrared radiation to space in the cool, subsiding equatorward moving air can reinforce this process. Surface friction, Ekman pumping, interior KH instability, geostrophic adjustment, wave-wave interaction, and heating/cooling will alter the instability.

4. Baroclinic Life Cycles

Zonal mean zonal winds maximize near 35 latitude and the tropopause. Idealized linear baroclinic life cycle experiments give poleward eddy heat fluxes that maximize in the lower troposphere, as observed. But the linear eddy momentum fluxes are very weak and do not exhibit a strong tropopause maximum as in observations. Numerical solutions at finite amplitude get the momentum fluxes right, because then the subtle phase relationship yielding nonzero u'v' is possible.

The pattern of momentum flux convergence acts to shift the jet poleward, while the Hadley circulation keeps trying to reinforce a strong subtropical jet. The breakdown of the subtropical westerly jet into baroclinically unstable waves is the primary mechanism for the hemispheric overturning Lagrangian transport of momentum and constituents in the troposphere. The energy pathway is: zonal mean available potential energy -> eddy kinetic and potential energy -> zonal mean kinetic energy. A Rossby wave breaking event will shift the jet poleward, reducing the temperature gradient.

Baroclinic wave growth time scales are as small as half a day in the storm tracks over the east coasts of the continents in winter. These waves grow, propagate poleward and eastward, then decay near the end of the jet streak.

V. Other Instabilities

1. General

Shear, surface tension, temperature, and rotation each have their own types of instability. A nonlinear instability requires a finite amplitude perturbation to overcome an initial resistance, while a linear instability does not. Linear instability is assessed for a basis set of distinct modes, for each mode separately. In reality, a disturbance may grow fastest because it is largest. Linear instability analysis does not do a good job of predicting finite-amplitude effects, such as changing the mean flow or constituent distributions. Numerical calculations are generally needed for that. An example is the fact that a linear baroclinic instability analysis gets the poleward heat flux right, but gets the momentum fluxes wrong.

2. Inertial Instability

Holton section 2.3 and 9.1
Kundu 11.5
Infinitesimal displacement, inviscid, environment is unaffected.

Assess if displaced parcel will accelerate.
Note the symmetry for convective and inertial instabilities.
A non-zero imaginary part of the frequency implies acceleration.
If it's zero then there will be stable oscillations (at the buoyancy or inertial period).
If circulation or angular momentum *decreases* outward from the rotation axis, then it is inertially unstable.
This inviscid criterion was calculated by Rayleigh in 1888.
Taylor in 1923 calculated the viscous case and compared it favorably to laboratory results for Couette flow between two rotating cylinders, prompting some to delight in the excellence of the Navier-Stokes equations, gloriously emancipated from pedestrian concerns of linear stress/strain and failure of the continuum at small scale.
Friction can stabilize weakly unstable distributions of potential temperature or angular momentum.

The criterion can be derived for a displacement delta y in the presence of a geostrophic flow by estimating u at delta y and u_g at delta y and comparing them.
If u > u_g there, then angular momentum decreases radially outward and it is inertially unstable.
This criterion applied to the earth is f (f+zeta) < 0, where zeta is relative vorticity.
Alternatively, f P < 0, where P is Ertel's PV, can be used to describe either convective or inertial instability.
P can be near zero in convection, near the equator, and in strong horizontal shear.

Cross-equatorial flow will advect anomalous P into the other hemisphere, rendering the air inertially unstable. Inertial instability is a primary process by which air crosses the equator and runs into a geostrophic regime. The ITCZ, mesospheric "pancake structures", poleward surges, and enhanced mixing in the Rossby wave breaking process are examples of phenomena attributable to inertial instability. As either inertial or convective stability evolves, overturning circulations imply the other type, so often both are occurring.

3. Wave Instabilities: The Normal mode method
Instabilities occur in the presence of inflection points interior to the fluid, including a jet, wake, interior shear, and free convection. Viscosity tends to stabilize normal mode instabilities.

Basic state flow satisfies equations. Add a small perturbation in a spatial pattern (as opposed to a single parcel being displaced).
If the imaginary part of the frequency or phase speed is non-zero then instability is occurring. This is an eigen problem, where each mode has a spatial pattern (eigenfunction) and phase speed (eigenvalue).

Recipe:
1. Simplify physics for the problem at hand
2. 2D streamfunction
3. Inviscid
4. Linearize
5. Assume sinusoidal form

6. Convert N first order differential equations to a single one of order N.
7. Apply boundary conditions

V. The Taylor-Goldstein equation, which results from considering perturbations in a linear vertical shear environment, has several interpretations: vertical structure equation, $Ri < 1/4$ criterion, emanation of wave energy from a critical surface, and dispersion relation (or index of refraction relation).

4. Kelvin-Helmholtz instability

Kelvin-Helmholtz instability is quite common in stratified flow with shear. If shear is strong enough for $Ri < \frac{1}{4}$ then an infinitesimal sinusoid will grow with time, break, and mix, redistributing the original distribution of mass and speed. After mixing, the center of mass is raised and the shear is reduced. Thus, kinetic energy is converted to potential energy with this process, which is sporadic and nearly ubiquitous, (but neither omnipresent, omniscient, or omnipotent). [Adherants of the Cult of Ubiquitous Gravity Waves and those of the Church of Potential Vorticity can find common ground in the Rossby adjustment problem.]

GI Taylor (1915) conjectured that $Ri < 1/4$ is necessary for KH instability. Then in 1961 Miles proved it using an integral constraint on the Taylor Goldstein equation. The TG equation is first transformed, then multiplied by the complex conjugate of the streamfunction, and then integrated over the domain. This yields an integral equation which must be satisfied by the imaginary and real parts separately (two equations!). Examining the imaginary part, growth can only occur if $Ri < 1/4$. Then gravity wave breaking yields a "Kelvin's Cats Eye Pattern", with attendant mixing and redistribution of the basic state such that PE increases at the expense of initial KE.

5. Barotropic Instability

Barotropic instability converts kinetic energy in the mean flow to eddy KE, reducing initial flow differences. A Kelvin's Cats Eye pattern develops near the critical surface, involving irreversible mixing. For the wave to grow, phase surfaces must be tilted opposite to the basic state shear. If $U_y > 0$ the wave can grow, if it fluxes momentum southward to where the mean flow is slower. This reduces kinetic energy of the zonal flow, and the wave grows at its expense. At the end of a baroclinic wave's life cycle the structure is vertically aligned (barotropic) and the wave decays by feeding energy back into the mean flow at a higher latitude. There the eddy momentum flux is into the jet, having the same sign as the shear on both sides of the jet.

When the above problem is cast in (x,y) instead of (x,z), with U_y instead of U_z, and rho=constant, the TG equation is obtained again. Multiplying by the complex conjugate, integrating across the channel from y_1 to y_2, the imaginary part of the integral equation gives the Rayleigh criterion, where growth can only occur if the curvature switches sign in the domain. This is Rayleigh's inflection point criterion (1888). It is a necessary condition for instability, but a proof of a sufficient condition has yet to be discovered! Maybe there isn't one, because when this criterion is satisfied, sometimes there are disturbances, sometimes not.

In 1950 Fjortoft used a different transform on the TG equation in y to show that vorticity must be a maximum in the interior of the flow (near the inflection point) and not at the boundaries for waves to grow.

In the 1950s Kuo included the Coriolis parameter, yielding the Rayleigh-Kuo criterion for barotropic instability, that beta-Uyy must switch sign in the domain. Equivalently, Py must switch sign in the domain. This can occur more easily near the poles, where beta -> 0. Otherwise one needs a pretty strongly curved jet. This is commonly the case in the tropospheric easterly jet over Africa in the northern summer. It is barotropically unstable and generates easterly waves (westward travelling synoptic scale Rossby waves which propagate by conserving absolute vorticity, with *relative* flow through them from west to east).
In the trade winds, weather variability is controlled by these easterly waves, much as our midlatitude weather is controlled by westerly Rossby waves.

In fact, most synoptic scale systems have growth components related to extracting energy from horizontal curvature (barotropic instability) and horizontal temperature gradients (baroclinic instability), as well as latent heating.

6. Howard's semicircle theorem
states that for wave growth the real phase speed must lie between the minimum and maximum flow speed. The disturbance emanates from a critical surface for KH, barotropic, or baroclinic instability in the interior of the fluid. Further, the growth rate is proportional to the difference between the maximum and mimimum wind speed. If the shear is stronger, then more energy can be extracted.