Wave-CISK in Mass Flux Form

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ABSTRACT

A reformulation of wave-CISK shows that gravity waves generated by the divergence of cumulus mass fluxes are responsible for the forcing of further convection. When downdrafts are included, a new, non-propagating instability arises. This mode has a growth rate much larger than the usual, propagating wave-CISK mode, but requires the downdraft mass flux to exceed a certain critical value. The nonpropagating mode apparently corresponds to air-mass thunderstorms, whereas the propagating mode suggests long-lived convection. The two modes respond very differently to wind shear.

1. Introduction

The CISK hypothesis (Conditional Instability of the Second Kind) in simplest terms states that cumulus convection is generated by convergence in the planetary boundary layer, and that the resulting heating of the troposphere, in turn, forces additional boundary layer convergence. The result is a self-excit ing disturbance or instability.

In the original application of the CISK hypothesis to hurricane dynamics, Charney and Eliassen (1964) and Ooyama (1964) envisioned the boundary layer convergence to be due to frictional effects in a region of cyclonic vorticity. This process is here called frictional CISK to distinguish it from wave-CISK, in which the convergence is induced by inviscid wave action (Yamasaki, 1969; Hayashi, 1970; Lindzen, 1974).

The author (Raymond, 1975, 1976) showed that wave-CISK has promise in explaining the behavior of certain severe convective systems. This was in spite of the use of a crude cumulus parameterization.

Parameterization continues to be the Achilles' heel of wave-CISK. This paper introduces a way of handling cumulus parameterization that appeals to the intuition and thus makes wave-CISK more understandable. By replacement of the total vertical velocity with the adiabatic vertical velocity, the main source term moves from the thermodynamic equation to the continuity equation. Mass sources and sinks associated with convective mass fluxes drive the waves responsible for producing convergence in the boundary layer.

An idealized mass flux profile in a windless environment of constant stability is used as a vehicle for exploring the characteristics of the new model. Various elements of a more realistic model are then tried in sequence, starting with lag effects in the updraft, and progressing through downdrafts and wind shear.

Downdrafts have long been thought to play an important role in the maintenance and propagation of convective systems. The mass flux formalism introduced here allows easy incorporation of downdrafts into the cumulus parameterization. I find that downdrafts generate a new type of wave-CISK instability which, in contradistinction to the usual propagating mode, is stationary in windless situations. This advecting mode has a large growth rate insensitive to scale, but occurs only when the downdraft exceeds a critical strength.

Wind shear is also known to have a strong effect on the characteristics of convective systems. I show that the propagating modes of wave-CISK tend to move against the shear, while the advecting modes move with the wind at some level.

The strengths and deficiencies of the present model, as well as its applicability to real situations, are discussed in Section 7.

2. Mass flux form of equations

For simplicity, I begin with the hydrostatic, Boussinesq, inviscid, nonrotating equations with dependent variables $u$ and $w$ the horizontal and vertical wind fields, $\pi$ the pressure perturbation divided by the mean density, and the buoyancy $\eta$, defined as the fractional potential temperature perturbation times the acceleration of gravity.

Dividing the horizontal wind into ambient and perturbation quantities $u = V(z) + v$, I lump all nonlinear and inhomogeneous terms into source functions on the right side of the equations:

$$\frac{\partial v}{\partial t} + \nabla v + \frac{dV}{dz} w + \nabla \pi = F,$$  

(1)
\begin{align}
\frac{\partial \pi}{\partial z} + \eta = 0, \quad \text{(2)} \\
\frac{\partial \eta}{\partial t} + \nabla \cdot \nabla \eta + N^2 w = H, \quad \text{(3)} \\
\nabla \cdot \mathbf{v} + \frac{\partial w}{\partial z} = 0. \quad \text{(4)}
\end{align}

Here \( H \) and \( F \) represent sources of buoyancy and momentum, and are given by
\begin{align}
F &= -\nabla \cdot (\mathbf{v} w) - \frac{\partial}{\partial z} (w \mathbf{v}), \quad \text{(5)} \\
H &= C - \nabla \cdot (\psi \eta) - \frac{\partial}{\partial z} (w \eta). \quad \text{(6)}
\end{align}

In (1)–(6), \( N(z) \) is the ambient profile of Brunt frequency, \( C \) the buoyancy source due to condensation minus evaporation, and \( \nabla \) the horizontal gradient operator.

I now partition the vertical velocity into the adiabatic vertical velocity \( w_a \) and a quantity \( M \), defined as
\[ M = \frac{H}{N^2}. \quad \text{(7)} \]

Eliminating \( w \) in favor of \( w_a \) in (1)–(4), I move the effects of the buoyancy source from the buoyancy equation to the continuity and momentum equations:
\begin{align}
\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{u} + \frac{d \mathbf{V}}{dz} w_a + \nabla \pi = \mathbf{F} - M \frac{d \mathbf{V}}{dz}, \quad \text{(8)} \\
\frac{\partial \eta}{\partial t} + \nabla \cdot \eta + N^2 w_a = 0, \quad \text{(9)} \\
\nabla \cdot \mathbf{v} + \frac{\partial w_a}{\partial z} = - \frac{\partial M}{\partial z}. \quad \text{(10)}
\end{align}

Outside of active convection, condensation and turbulent heat fluxes are small. Thus \( w_a \approx w \) there. Inside convection \( w_a \) is that imaginary vertical velocity required to make the linearized thermodynamic equation adiabatic, based on the existing \( V \) and \( \eta \) fields. In contrast to \( w \), \( w_a \) is likely to be the same order of magnitude inside and outside of convection. This follows from (9) because \( \eta \) values inside clouds do not greatly exceed those in the environment. As a consequence, \( |w_a| \) will usually greatly exceed \( |w| \) in cloud.

The area-integrated value of \( M \) may be obtained observationally. Integrating (10) over the area of a convective system and using the divergence theorem results in
\[ -\frac{d}{dz} \int M dA = \oint \mathbf{v} \cdot \mathbf{n} ds \quad \text{(11)} \]
where \( w_a \) has been neglected in comparison with \( M \) in accord with the above arguments. The line integral

is around the periphery of the convective region, and \( \mathbf{n} \) is the unit normal to the periphery.

To the extent that (11) is valid, \( M \) is the cumulus mass flux. I therefore denote (8)–(10) the mass flux form of the equations of motion. Ooyama (1964, 1971) first wrote the governing equations in this form.

Measurements of the right side of (11) have been made using sounding arrays (Fankhauser, 1969; McNab and Betts, 1978), doppler radar (Miller et al., 1982; Frank and Foote, 1982), and circumnavigating aircraft (Foote and Fankhauser, 1973; Raymond and Wilkening, 1982). The area averaged mass flux forcing is thus in principle available from observation without penetration of active convection.

The basic idea of wave–CISK is that upward moving air at the level of free convection (LFC) \( z = b \) is the source of cumulus mass flux, i.e.,
\[ M(z) = \frac{w_u}{b} M_a(z), \quad \text{(12)} \]
where
\[ w_u = \max(w|_{z=b}, 0) \quad \text{(13)} \]
is the vertical wind at the LFC, if positive, and zero otherwise, and where \( M_a(z) \) is a normalized mass flux profile defined so that \( M_a(b) = b \).

Cumuli do not respond instantaneously to upward motion, and Davies (1979) showed that lag effects can be important to wave–CISK. A generalization of (12) that encompasses this possibility is
\[ M = \frac{1}{b} \int_{-\infty}^{t} dt' \int d\mathbf{A} B(\mathbf{r} - \mathbf{r}', z - t') w_a(\mathbf{r}', t'), \quad \text{(14)} \]
where \( B(r, z, t) \) is the mass flux distribution resulting from an impulse of upward motion at \( (r, t) = 0 \), where \( \mathbf{r} \) is the horizontal position vector.

For \( B = M_a \delta(r) \delta(t) \), the simple form (12) is recovered. If the updraft is lagged by time \( \tau \) and advances at velocity \( V_a \) the form \( B = M_a \delta(\mathbf{r} - V_a \tau) \delta(t - \tau) \) is appropriate. A distribution of lags is even possible:
\[ B = \int_{0}^{\infty} D(z, \tau) \delta(\mathbf{r} - V_a \tau) \delta(t - \tau) d\tau, \quad \text{(15)} \]
where \( D \) is the lagged mass flux profile. Substitution in (14) yields
\[ M = \frac{1}{b} \int_{0}^{\infty} D(z, \tau) w_a(\mathbf{r} - V_a \tau, t - \tau) d\tau. \quad \text{(16)} \]

An expression for the momentum source function \( \mathbf{F} \) similar to (16) would complete the model. However, the proper form of \( \mathbf{F} \) is very uncertain at this time (LeMone, 1983). Raymond (1975) showed, using a scale analysis, that the effects of \( \mathbf{F} \) on wave-CISK modes was likely to be small compared to the buoyancy source. Therefore \( \mathbf{F} \) is neglected for the purposes of this paper.

Eq. (16) closes the system in a linear fashion and makes an analysis in terms of eigenfunctions possible.
I let \( \eta = \tilde{\eta} \exp[ik(x - ct)] \), etc., and make the customary assumption that \( \tilde{w}_u = e^{i\omega}(b) \), where Lindzen (1974) showed that the duty cycle \( \epsilon = 1/2 \). This approximation is necessary because convective forcing results only from convergence. Divergence does not introduce negative forcing. The duty cycle approximation artificially spreads the forcing over the full wave with correspondingly reduced magnitude. No generality is lost by assuming a plane wave in the \( x \)-direction, since the \( x \)-axis may be arbitrarily aligned.

Defining the auxiliary variables \( d = \nabla \cdot v \) and \( \chi = \nabla \pi \), eliminating \( \eta, \pi \), and \( v \) from (2) and from (8) to (10), I obtain

\[
\frac{\partial \tilde{\chi}}{\partial z} - \frac{N^2}{c - V} \tilde{\chi}_a = 0, \tag{17}
\]

\[
\frac{\partial \tilde{w}_u}{\partial z} + \tilde{d} = -\frac{\partial \tilde{M}}{\partial z}, \tag{18}
\]

\[
d = \frac{1}{c - V} \left[ \frac{dV}{dz} \tilde{w}_u + \tilde{\chi} + \tilde{\chi}_a \frac{dV}{dz} \right]. \tag{19}
\]

The plane wave assumption reduces (16) to

\[
\tilde{M} = \left[ \frac{e^{i\omega}(b)}{b} \right] \int_0^\infty D(z, \tau) e^{i(kz - \omega \tau)} d\tau \]

\[
\left[ \frac{e^{i\omega}(b)}{b} \right] M_a(z) = -\left( \frac{e^{i\omega}(b)}{b} \right) \int_0^b \tilde{d} d\tau, \tag{20}
\]

where the last step results from the original continuity equation (4); the \( V, V_a \), and \( \chi \) are, respectively, the \( x \)-components of \( V, V_a \), and \( \chi \).

As a final approximation, it is assumed that the convergence at \( z = 0 \), \( d(0) = \tilde{d} \), adequately represents the average convergence in the layer \( 0 < z < b \). This assumption makes the solution of (17)–(20) easier, and though not desirable where quantitative results are required, is justified here for clarity's sake. Equation (20) thus becomes

\[
\tilde{M} = -e^{i\omega}(b) \tilde{d}, \tag{21}
\]

where

\[
M_a(z) = \int_0^\infty D(z, \tau) e^{i(kz - \omega \tau)} d\tau. \tag{22}
\]

3. Fourier space and configuration space

I now examine the solutions of (17), (19) and (21) in the case in which \( N \) is constant and \( V = 0 \). Under these conditions the equations simplify to

\[
\frac{\partial \tilde{d}}{\partial z} - \alpha^2 \tilde{d} = -\alpha^2 \frac{dM_n}{dz}, \tag{23}
\]

which has the solution that vanishes at \( z = \infty \),

\[
\tilde{d} = A e^{-\alpha z} + \frac{\epsilon d}{2} \int_0^\infty \left[ \frac{d}{dz} M_a(z) \right] e^{-\alpha z} d\tau', \tag{24}
\]

where \( A \) is a constant to be determined and \( \alpha = iN/c \).

In order to make the vertical velocity vanish at \( z = \infty \), the condition

\[
\int_0^\infty \tilde{d} d\tau = 0 \tag{25}
\]

must hold. This leads to a determination of \( A \):

\[
A = (\epsilon d d/2) \int_0^\infty \frac{dM_n}{dz} e^{-\alpha z} d\tau. \tag{26}
\]

Evaluating (24) at \( z = 0 \) canceling \( d \), yields the dispersion relation:

\[
1 - \alpha \epsilon \int_0^\infty \frac{dM_n}{dz} e^{-\alpha z} d\tau = 0. \tag{27}
\]

Fig. 1 shows mass flux profiles used in this paper, with \( M_a(z) \) representative of the normalized mass flux in an updraft. Uniform inflow occurs up to the LFC at \( z = b \), above which there is nonentraining ascent to \( z = a \), where the entire flux is detrained. Substitution of \( M_n = M_a \) into (27) yields

\[
1/\epsilon - 1 + e^{-\alpha b} + ab e^{-\alpha a} = 0. \tag{28}
\]

Eq. (28) shows the complex phase speed \( c \) predicted by (28) as a function of \( b \) for \( \epsilon = 1/2, N = 1 \) and \( a = 1 \). The latter two conditions set the time and length scales respectively. If the actual Brunt frequency is \( 10^{-2} \) s\(^{-1}\) and the outflow occurs at \( z = a = 10 \) km, the scaling velocity is 100 m s\(^{-1}\).

Eq. (28) has multiple solutions corresponding to the fundamental mode and higher harmonics. The first three modes are shown in Fig. 2. Growth rates are very sensitive to the LFC, but phase speeds are relatively insensitive. It is interesting that the fundamental mode does not always dominate in growth rate. As \( b \) becomes smaller, higher modes successively become the fastest growing. The hydrostatic approximation makes \( c \) independent of \( k \), resulting in growth rates proportional to \( k \).

As formulated in this paper, the computed flow patterns are the response of the adiabatic, linearized gravity

![Fig. 1. Mass flux profiles used in this paper: updrafts (left); downdrafts (right).](image-url)
can be accounted for by placing a symmetric image mass source below the surface, but the waves from this source are omitted for clarity.

The arrows in Fig. 3 show the wind resulting from the source and its image. The effect for an observer at a fixed point on the surface is an alternation in the wind toward and away from the source.

A consequence of the oscillations in surface wind is a series of progressive waves of surface convergence and divergence moving away from the source. Differentiation of (33) with respect to $x$ yields this divergence pattern

$$d = \frac{1}{2\pi x^2} (\phi \sin \phi - \cos \phi),$$

where $\phi = zt/x$.

Fig. 4 shows a plot of $2\pi x^2d$ versus $\phi$. Regions of convergence occur at local minima of $\phi$. Excluding the minimum at the origin which will be shown to be of no consequence, the first two minima are at $\phi = 5.1$ and $11.1$.

If the source is located at an elevation $z = \alpha = 1$, these regions of convergence move away from the origin with speeds $x/t = z/\phi = 1/\phi = 0.196$ and $0.090$. Comparison with Fig. 2 shows that the first and second modes, respectively, move at speeds very close to those of the first two convergence waves.

I propose the following mechanism to explain this correspondence: The convergence driving convection
waves. It does, however, cancel the convergence at $\phi = 0$.

Since the unstable mode described in this section depends on the divergence of the updraft mass flux for its existence, it is called the updraft mode, to distinguish it from other modes to be discussed later.

4. Lagged updraft

Three significant criticisms of the idealized model of Section 3 pertain to the fact that wind shear, downdrafts and convective lag effects are neglected. In this section the effect of lagged updrafts are examined. Downdrafts and shear are considered in the next two sections.

Davies (1979) developed a model of convective lags in wave–CISK in which all the effects of convection are uniformly lagged by some time interval. The result was a short wavelength cutoff for wave–CISK modes.

Davies’ assumption of uniform lag is less applicable to the small scale, where the time required by a convective element to evolve through its life cycle may be significant. I therefore propose a model is propose in which convective “bubbles” form instantly at the LFC in response to convergence, but require finite time to rise through the atmosphere.

A simple realization of this model is manifested in the normalized mass flux profile

$$M_\phi(z) = \begin{cases} 
  z, & z < b \\
  b \exp[i\mu c(z - b)], & b < z < a \\
  0, & z > a,
\end{cases}$$

(37)

where the lag parameter is defined to be $\mu = k/W$, $W$ being the (constant) ascent rate of convective bubbles through the atmosphere.

Eq. (37) arises from applying (22) to an assumed lagged mass flux profile of the form

$$D(z, \tau) = \begin{cases} 
  zd(\tau), & z < b \\
  b[d(\tau) - (z - b)/W], & b < z < a \\
  0, & z > a,
\end{cases}$$

(38)

For $z < b$, the mass flux is the same as in the idealized model of Section 3. However, for $b < z < a$ a pulse of mass moves upward at speed $W$, reaching elevation $z$ after a time lag $(z - b)/W$. Zero advection velocity $V_a$ is assumed.

At long wavelengths $k$, and hence $\mu$, goes to zero and (37) reduces to the idealized case.

Substitution of (37) into (27) yields the dispersion relation:

$$\left(1 - \frac{1}{\epsilon} \right) \left( \mu + \alpha^2 \right) + \left( \mu + \alpha^2 + \alpha b \mu \right) e^{-\alpha b}$$

$$+ \frac{\alpha^2 b e^{-\alpha a} - \frac{\mu(a - b)}{\alpha}}{\alpha} = 0.$$ 

(39)
Fig. 5 shows solutions to (39) for the first two wave-CISK modes, with $b = 0.3$, $a = 1$, $\epsilon = 0.5$, and illustrates how the complex phase speeds change when lag is introduced. Lag effects reduce $\text{Re}(c)$ for both modes; $\text{Im}(c)$ of the fundamental increases up to $\mu = 70$, while that of the first harmonic decreases. After $\mu = 70$, $\text{Im}(c)$ for the fundamental also undergoes a slow decrease (not shown).

An estimate of $\mu$ illustrates the significance of this lag effect. With the previous scaling, a wavelength $2\pi/k = 30$ km and a bubble rise speed of $10$ m s$^{-1}$ yield $\mu = 20$, which according to Fig. 5 is a significant lag.

This type of lagged updraft yields results that are quite different from those of Davies (1979). The short wavelength cutoff found by Davies simply does not occur. Introduction of an additional lag between convergence and bubble initiation would presumably restore this effect, however.

Bolton (1980) showed that propagating wave-CISK modes must exhibit a phase shift between the diabatic heating and the vertical velocity. Since the heating is proportional to $M$, this reduces to a phase difference between $M$ and $w$.

In the present wave-CISK model $M$ and $w$ are in phase at the LFC, which implies that one or the other of these variables must exhibit a phase shift with height. In an unlagged updraft this devolves upon $w$, and hence $w_d$. An updraft lagged in the sense of this section results in a decrease of phase speed and an increase of growth rate, at least for the fundamental mode. This suggests from Bolton’s analysis that lagging the updraft results in a decrease in the phase difference between $M$ and $w$ above the level of free convection.

5. Downdrafts

In this section, I ignore lag in the updraft but explore its effect on the downdraft. This makes it possible to account for the displacement of the downdraft (in space, time, or both) from the updraft characteristically observed in thunderstorms. Downdrafts presumably take place when precipitation evaporates in dry, midlevel air, resulting in a downward mass flux into the boundary layer. Fig. 1 shows an idealized downdraft mass flux profile $M_d$ representing this process. The strength of the downdraft is given by the dimensionless parameter $\gamma$.

I assume a lagged mass flux profile of the form

$$D(z, \tau) = K \left[ M_d(z) \delta(\tau) - \frac{M_d(z)e^{-\tau \tau_0}}{\tau_0} \right],$$

where the constant $K$ is adjusted to produce the required normalization $M_d(b) = b$. The first term in (40) represents an unlagged updraft whereas the second represents the downdraft with an exponential lag distribution. The mean lag is $\tau_0$.

Substituting (40) into (22) and invoking the normalization, I obtain

$$M_d(z) = M_d(z)[1 - i\mu(c - V_a)] - M_d(z) \left( \frac{1}{1 - i\mu(c - V_a)} \right),$$

where the lag parameter is now $\mu = k\tau_0$, and $M_u$ and $M_d$ are as specified in Fig. 1.

For the case $V = 0$ and $N = 1$, the dispersion relation is obtained by substituting (41) into (27):

$$(1/e + e^{-\alpha b} - 1)(1 - \gamma - i\mu c) - \gamma(1 - e^{-\alpha b})e^{-\alpha b} + \alpha b(1 - i\mu c)e^{-\alpha b} = 0.$$  

The advection velocity $V_a$ is here associated with the horizontal movement of precipitation from formation to fallout regions, and is set to zero, consonant with the zero ambient wind.

The strength $\gamma$ of the downdraft mass flux can be related to the precipitation efficiency of the convective system. I define the precipitation efficiency $e$ as the ratio of the condensed water reaching the surface to the total condensed water. Relating condensation to diabatic heating and recalling that $H = N^2 M$, I find

$$e = 1 - \int_0^\infty N^2 M_d dz \int_0^\infty N^2 M_d dz$$

which results in

$$\gamma = (1 - e)(\frac{a}{b} - \frac{1}{2})$$

for $N = 1$ and $M_d$ and $M_u$ as shown by Fig. 1.
For low precipitation efficiencies, \( \gamma \) can actually exceed unity, which means that in the unlagged case the downdraft mass flux is actually greater than the updraft flux in the boundary layer.

Figs. 6–8 show the complex phase speed \( c \) as a function of rainfall efficiency for three different values of the lag parameter \( \mu \). For all cases \( a = 1 \) and \( b = 0.3 \) which yields a critical rainfall efficiency \( e = e_c = 0.647 \), defined as that \( e \) for which \( \gamma = 1 \). This is shown by the vertical line in Fig. 6.

As might be expected, behavior is rather singular near the critical rainfall efficiency. Examining the unlagged case first (Fig. 6), I note that efficiencies well in excess of the critical value yield modified updraft modes with phase speeds near the no downdraft value. The growth rate of the fundamental mode is enhanced, while the harmonics are suppressed. For this reason only the fundamental is illustrated. As the rainfall efficiency approaches the critical value from above, the growth rate greatly increases, and the phase speed goes to zero.

For efficiencies less than critical, a propagating mode again occurs, but with phase speed and growth rate which differ from the case in which \( e > e_c \). This mode is similar to the updraft mode except that it is primarily driven by the downdraft. It is therefore named the downdraft mode.

Fig. 9 supports this interpretation. Shown are the real and imaginary parts of the phase speed of the downdraft mode for the extreme case \( e, \mu = 0 \). For comparison with \( \text{Re}(c) \), the dotted line shows the speed of the first divergence wave caused by an impulsive mass sink at the mean elevation of the downdraft inflow, \( z = 1.5 b \). From Section 3, this is given by \( x/t = z/\psi = 1.5 b/5.1 = 0.294 b \) for the fundamental. This wave is now divergent rather than convergent because the source term is reversed in sign.

As Fig. 4 indicates, a region of convergence travels ahead of the downdraft (i.e., has a smaller value of \( \psi \)). This convergence is responsible for producing the convection that generates the downdraft.

A new mode appears when \( \mu > 0 \) and \( e < e_c \). This is labeled the advecting mode in Figs. 7 and 8, and in agreement with its name, it doesn’t propagate, but moves with the ambient wind. As Fig. 7 shows, the advecting mode branches off from the fundamental propagating mode near the critical rainfall efficiency and attains rather large growth rates for \( \mu = 1 \).

For \( \mu = 10 \) (Fig. 8) the advecting mode is no longer connected to the propagating mode, which is now continuous through the critical rainfall efficiency with no region in which \( \text{Re}(c) = 0 \).

To understand the origin of the advecting mode, note that for purely imaginary \( c \) with \( \text{Im}(c) < 0.1 \), \( b \)
or sink aloft moving in phase with its self-generated surface convergence or divergence field. Advection
modes do not propagate, but they grow rapidly in time. Could an exponentially growing, stationary
source aloft similarly produce convergence?

I answer this question by convolving (33) with a point source aloft of exponentially growing strength
\( \exp(\lambda t) \). The resulting velocity field due to this mass source is:

\[
v = \frac{1}{2\pi x} \int_0^\infty \cos \left( \frac{zt}{x} \right) e^{\lambda(t-t_0)} dt = \frac{e^{-\lambda x}}{2\pi \lambda^2 x^2 + z^2}
\]

(47)

which implies divergence at \( x = 0 \) equal to

\[
d|_{x=0} = \frac{\partial v}{\partial x} \bigg|_{x=0} = \frac{\lambda e^{-\lambda x}}{2\pi z^2}.
\]

(48)

A stationary, but intensifying source aloft thus generates surface divergence directly beneath it. The an-
vil level outflow therefore works against its own perpetuation by this mechanism. However, intensifying
inflow aloft, such as that feeding the downdraft, causes surface convergence, which amplifies the con-
vection, the rain production, the downdraft, and consequently the inflow. Due to the near balance of up-
draft and downdraft below \( z = b \), the low level mass flux divergence isn’t important. The advecting mode
is thus a consequence of positive feedback between surface convergence and convergence aloft.

Recalling that \( \mu = k\tau_0 \), (46) may be rewritten

\[
\mu \Im(c) = \tau_0 \Im(\omega) = (e_c - e)(a/b - \frac{1}{2})
\]

(49)

which shows the growth rate of the advecting mode to be approximately independent of scale. This is con-
formed in Fig. 10 in which both the approximation (49) and the exact solution to (42) are plotted versus
\( \mu \) for the case \( b = 0.3 \) and \( e = 0 \). The exact solution shows slight enhancement in growth rate near \( \mu = 9 \) due to the other terms in the dispersion relation. Also shown in Fig. 9 is the dispersion relation for the
downdraft mode, which is seen to have a much smaller growth rate than the advecting mode.

For rainfall efficiencies greater than the critical value, the advecting mode disappears and the updraft
mode dominates. Fig. 11 shows the fundamental mode dispersion relation for \( e = 0.7 \) and 1. The \( e = 1 \) case has no downdraft and therefore \( c \) exhibits no \( \mu \) dependence. Comparison of the two shows how the
downdraft enhances the growth rate at small \( \mu \) (and hence long wavelength) without greatly affecting
the propagation speed.

6. Shear

When shear is present, the simplified governing equation (23) no longer applies. However (17)–(19)
may be directly integrated on a computer from the surface up, repeating this process with different com-

![Diagram of phase speed for the downdraft mode](image-url)
plex phase speeds until an upper radiation boundary condition is satisfied.

I assume constant stability and wind for \( z \gg 1 \), with \( N \) and \( V \) chosen there to be continuous with the respective profiles for \( z \ll 1 \). Under these conditions the radiation boundary condition is easily shown to be

\[
\tilde{\chi} - iN\tilde{\psi} = 0 \quad (50)
\]

for \( z \gg 1 \) as long as \( M = 0 \) there.

The above scheme was implemented with \( N(z) = 1 \) and \( V(z) = C_1(z - \frac{1}{2}) + C_2(z - \frac{1}{2})^2/2 \) for \( z \ll 1 \). The normalized mass flux including downdrafts as given by (41) is used with \( a = 1 \) and \( b = 0.3 \), and (21) is invoked to link \( M \) and \( M_n \). The precipitation is assumed to advect with the wind at mid-levels whence \( \nabla_u = V(a/2) = V(1/2) \).

The phase speeds and growth rates for various mass flux configurations and wind profiles are summarized in Figs. 12 and 13. These are now discussed.

\[ \text{a. The updraft mode} \]

In the limit \( \epsilon = 1 \) the solutions revert to the pure updraft mode of Section 3. The squares in Figs. 12 and 13 show how this mode behaves in shear. The fundamental and the first harmonic are included.

The main result is that shear imposes a strong selective effect in that modes propagating down-shear are suppressed as well as most up-shear modes. Only the fundamental mode moving up-shear survives shears exceeding 0.2. This mode is in fact greatly enhanced in growth rate by the shear.

\[ \text{Since the updraft mode results from low level convergence produced by waves originating at high levels, a WKB analysis of wave propagation in a sheared atmosphere sheds light on the selection mechanism. Eliminating d and } \chi \text{ from (17)–(19) and setting } M \]

\[ \text{fig. 10. Dispersion relations for the downdraft and advecting modes. The advecting mode has Re}(c) = 0. \text{ Since } \mu = kr_\mu, \text{ Im}(c) \text{ is proportional to the growth rate. The dotted line is an approximation to the advecting mode growth rate discussed in the text.} \]

\[ \text{fig. 11. Dispersion relations for the updraft mode with } \epsilon = 0.7 \text{ and } 1. \text{ Only the fundamental modes are shown.} \]

\[ \text{fig. 12. Effect of constant shear. Four wind profiles and the resulting phase speeds are shown for 1) updraft modes with } \epsilon = 1 \text{ (squares); 2) downdraft modes with } \epsilon = 0, \mu = 0 \text{ (circles); 3) advecting modes with } \epsilon = 0, \mu = 10 \text{ (triangles); and 4) enhanced modes with } \epsilon = 1.5 \text{ (diamonds). The position on the velocity axis gives Re}(c) \text{ for each of these symbols, while the vertical scale on the right gives Im}(c). \text{ Note the discontinuity in this scale; } C_1 \text{ and } C_2 \text{ are coefficients used in the text to define the wind profile, and LFC refers to the level of free convection } b.} \]
of \(|c - V|\) at the surface are attained on the concave side without a critical level occurring aloft. This is not possible on the convex side. 2) Because of the term proportional to \(d^2 V/dz^2\) in (52), \(p\) is larger on the concave side, other things being equal. As discussed before, this enhances the surface convergence.

b. Downdraft mode

As discussed in Section 5, the downdraft mode is the consequence primarily of waves radiated from the downdraft inflow region. Most of the arguments invoked for the updraft mode in shear therefore also apply to the downdraft mode. As Figs. 12 and 13 show, this mode (represented by circles) also propagates up-shear, against jets, and lacks critical levels. In the stronger wind cases the downdraft mode even takes on a phase speed very close to the speed of the most unstable updraft mode.

c. Advection mode

In a sheared environment the advection mode moves at a speed very close to the precipitation advection speed \(V_\alpha\). This is seen in Figs. 12 and 13 in which the advection mode with \(\mu = 10\) is represented by triangles. Recall that \(V_\alpha\) is set to the ambient wind at \(z = \frac{1}{6}\). Other choices of \(V_\alpha\) result in concomitant values of \(\text{Re}(c)\).

Other than through its effect on transport of precipitation, the wind profile has little impact on the advection mode. Growth rates are nearly independent of shear.

These results are easy to understand if the essential characteristic of the advection mode is assumed to be small boundary layer mass flux. This forces the denominator of (41) to approximately vanish, which results in

\[c \approx V_\alpha + i(\gamma - 1)/\mu.\]  

(55)

This generalization of the no shear equation (44) is in agreement with numerical results, which supports the above assumption.

d. Comparison with previous work

The earlier wave–CISK model of Raymond (1975) is very similar to the present model with downdrafts excluded and with an increased duty cycle \(\epsilon = 1.5\). This threefold enhancement of the duty cycle was justified empirically, and may to some extent simulate the effects of strong downdrafts.

Examination of the dispersion relation (28) shows that this increase in \(\epsilon\) changes the sign of \(1/\epsilon - 1\), which results in a fundamental change in the nature of the predicted modes. For no shear, the modes are nonpropagating with a very large growth rate, comparable to those of the advection modes. In fact, these modes work in a manner very similar to the latter—
the only difference is that the updraft inflow rather than downdraft inflow is responsible for creating surface convergence. This is possible because the threefold enhancement in the forcing generates a large growth rate, which is necessary for this mechanism to operate.

For constant shear (Fig. 12, diamonds) these enhanced modes have phase speeds quite different from the updraft and downdraft modes, but similar to the advecting modes. Curiously, the enhanced modes have speeds similar to the downdraft modes for the jet profile.

7. Discussion and conclusions

The main result of this paper is that there appear to be two basic types of wave-CISK modes with very different properties, namely, propagating and advecting modes.

Propagating modes fall into two subclasses, depending on whether they are driven primarily by updraft or downdraft mass flux divergences. However, both types share the following characteristics:

1. They propagate against the ambient shear and do not have critical levels.
2. For a jet-like wind profile, propagation against the jet is favored.
3. Maximum growth rate tends to be at short wavelengths.
4. Growth rates tend to be small.

In contrast, advecting modes have the following properties:

1. They move with the precipitation aloft as it is transported away from the updraft where it is formed.
2. Growth rates tend to be much larger than those of propagating modes.
3. Growth rate is almost independent of wavelength.
4. Advecting modes require the downdraft mass flux resulting from a particular updraft element to exceed the updraft mass flux at the level of free convection. However, for the instability to exist, the downdraft must lag the updraft.

Nonlinear effects ultimately force exponential growth to stop. For the propagating modes this is likely to result in a steady disturbance of large amplitude. This appears to be impossible for the advecting mode for the following reason: If the growth rate goes to zero at some large amplitude, the downdraft mass flux catches up with and exceeds the updraft flux, resulting in divergence at low levels. Simultaneously, the convergence forced by the midlevel inflow shuts off, since this mechanism is dependent upon continued intensification. As a result, the disturbance decays.

This is reminiscent of the sequence of events occurring in the life cycle of an airmass thunderstorm as originally described by Byers and Braham (1949). Therefore, I suggest that the advective instability is the dynamical mechanism of such storms.

It is tempting to identify the propagating updraft and downdraft wave–CISK modes with long-lived convective disturbances such as tropical squall-lines (e.g., Zipser, 1969, 1977; Houze, 1977) and mesoscale convective complexes (Maddox, 1980; Bosart and Sanders, 1981). These disturbances often move faster than the wind at any level, as do propagating modes.

An apparent strike against this identification is the tendency of propagating modes to have their largest growth rates at short wavelengths. This contrasts with the large size often attained by long-lived disturbances. However, this may be an illusionary problem, because even though such systems are very large, most of the strong convection takes place in a narrow band along its advancing edge. Such a band would be quite adequately described by a short wavelength, propagating mode.

Short wavelengths invalidate the hydrostatic assumption. Unfortunately, it can be shown that nonhydrostatic terms are comparable to, and act in concert with the momentum source term. Thus, proper evaluation of nonhydrostatic effects requires consideration of momentum forcing for consistency. Since this has been omitted, nonhydrostatic effects have been neglected as well.

In a quantitative, as opposed to an illustrative model, many of the approximations imposed here need to be lifted. In particular, the simple relationship between mass flux and heating breaks down at low levels in precipitation. Approximating the mean divergence in a layer by its surface value is also questionable then. The Coriolis force may also be important for the larger disturbances.

Correction of the above flaws and use of environmental and mass flux profiles obtained from case studies will allow more direct comparison with observation.

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APPENDIX

Response of a Stratified Fluid to a Mass Impulse

A potential function $\phi$ may be defined such that

$$v = \frac{\partial \phi}{\partial x},$$  \hspace{1cm} (A1)

$$w_a = \frac{\partial^3 \phi}{\partial z \partial t^2},$$  \hspace{1cm} (A2)
\[ \eta = -\frac{\partial^2 \phi}{\partial z \partial t}, \]  
(A3) 

\[ \pi = -\frac{\partial \phi}{\partial t}. \]  
(A4) 

Equations (A1)–(A4) are consistent with (29)–(32), and it may further be shown that \( \phi \) obeys 

\[ \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = \delta(x) \delta(z) \delta(t) \]  
(A5) 

Attempts to solve (A5) by Fourier transform techniques lead to singular integrals. However, replacement of \( \delta(x) \) by 

\[ \Delta(x, \nu) = \frac{\nu}{\pi(x^2 + \nu^2)} \]  
(A6) 

leads to a problem that can be solved. Further, as \( \nu \to 0, \Delta(x, \nu) \to \delta(x) \).

Use of the Fourier transform in (A5) on \( x \) and \( t \) with \( \delta(x) \) replaced by \( \Delta(x, \nu) \) results in 

\[ \frac{\partial^2 \tilde{\phi}}{\partial z^2} + \frac{k^2}{\omega^2} \tilde{\phi} = \frac{-\delta(z)}{\omega^2} e^{-|k|\nu}, \]  
(A7) 

where \( k \) and \( \omega \) are respectively the \( x \) and \( t \) transform variables. The solution to (A7) with radiation away from the source for both positive and negative \( z \) is 

\[ \tilde{\phi} = \frac{\exp(-i|k|z/\omega - |k|\nu)}{(2i|\omega|k)}, \]  
(A8) 

I am unable to invert \( \tilde{\phi} \) directly to obtain \( \phi \), but both \( ik\tilde{\phi} = \tilde{\eta} \) and \( \partial \phi / \partial z \) can be inverted. This allows determination of every variable but \( \pi \). After inverting and letting \( \nu \to 0 \), I find 

\[ \nu = \frac{\cos(zt/x)}{(2\pi x)}, \]  
(A9) 

\[ \omega_a = \frac{\delta(x) \delta(t) z + z \cos(zt/x)}{(2|z|)} \]  
(A10) 

\[ \eta = -\frac{\sin(zt/x)}{(2\pi x)} \]  
(A11) 

for \( t \geq 0 \). For \( t < 0 \) all variables are zero.

REFERENCES


