Governing equations for a nonhydrostatic atmosphere
Conservation of Momentum

- The equations of motion for a cloudy atmosphere on a Cartesian framework can be written in tensor form:

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \delta_{i3} g + f_i
\]

Local acceleration

Coriolis acceleration

Gravity acceleration

advection

Pressure gradient acceleration

Molecular dissipation (we will ignore)

Permutation tensor:

\[
\delta_{i,j} = \begin{cases} 
1 & \text{if } i=j \\
0 & \text{if } i \neq j 
\end{cases}
\]

Kronecker Delta:

\[
\varepsilon_{i,j,k} = \begin{cases} 
0 & \text{if } i=j, \text{ or } j=k, \text{ or } i=k \\
1 & \text{if } i,j,k \text{ are } 1,2,3 \text{ or } 2,3,1 \text{ or } 3,1,2 \\
-1 & \text{if } i,j,k \text{ are } 3,2,1 \text{ or } 2,1,3 \text{ or } 1,3,2 
\end{cases}
\]

Spatial coordinate (x,y,z): (m)

Time: (s)

Velocity tensor: (m s⁻¹)

Gravity: (m s⁻²)

Pressure: (Pa)

Air parcel density: (kg m⁻³)
Air Parcel Density

- If we assume that all hydrometeors are moving at their terminal velocity, then

\[
\rho = \frac{\sum \text{mass}}{\text{Parcel Volume}} = \rho_d + \rho_v + \rho_l + \rho_i = \rho_m + \rho_l + \rho_i
\]

- The equation of state for this parcel is:

\[
p = \rho_m R_d T \left(1 + 0.61 q_v\right) = \rho_m R_d T_v
\]
Perturbation Form

• It is sometimes convenient (and perhaps even more accurate numerically) to remove the underlying mean hydrostatic balance from the equation

• Define a Reference State to represent this mean hydrostatic state. The reference state definition is arbitrary, but the closer it is to the model mean state, the smaller the perturbations. Since numerical error is proportional to the full gradient computed, error in computing the mean, hydrostatically balanced vertical pressure gradient dominates the error and yet we do not need to explicitly compute it if we remove it analytically first!
Reference State

• Assume it is defined to be:
  – A function of height
    \[ p_o = p_o(x_3); \rho_o = \rho_o(x_3); T_o = T_o(x_3) \]
  – Hydrostatic
    \[ \frac{1}{\rho_o} \frac{\partial p_o}{\partial x_3} = -g \]
  – Dry
    \[ q_o = 0; \rho = \rho_d \]
  – Obey equation of state
    \[ p_o = \rho_o R_d T_o \]
Also:

- We define dry potential temperature:

\[
\theta_o = T_o \left( \frac{p_{oo}}{p} \right)^{R_d / c_p}
\]
Subtract out hydrostatic base state

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_{j} u_{k} - \frac{1}{\rho} \left[ \frac{\partial (p_o + p')}{\partial x_i} + \left( \rho_o + \rho' \right) g \delta_{i3} \right]
\]

\[
= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_{j} u_{k} - \frac{1}{\rho} \left[ \frac{\partial p'}{\partial x_i} + \frac{\partial p_o}{\partial x_i} + \rho_o g \delta_{i3} + \rho' g \delta_{i3} \right]
\]

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_{j} u_{k} - \frac{1}{\rho} \left[ \frac{\partial p'}{\partial x_i} + \rho' g \delta_{i3} \right]
\]

\[
p'(x_1, x_2, x_3) = p(x_1, x_2, x_3) - p_o(x_3)
\]

\[
\rho'(x_1, x_2, x_3) = \rho(x_1, x_2, x_3) - \rho_o(x_3)
\]
Thermodynamic Relationships

\[ T = \theta \left( \frac{p}{p_{oo}} \right)^{R_d/c_p} \]

and so the equation of state becomes:

\[ p = \rho_m R_d \theta \left( \frac{p}{p_{oo}} \right)^{R_d/c_p} (1 + 0.61q_v) \]

so

\[ p^{1-R_d/c_p} = p^{\frac{c_p-(c_p-c_v)}{c_p}} = \frac{1}{p^\gamma} = \frac{\rho_m R_d \theta_v}{p_{oo}^{R_d/c_p}} \]

Equation of State with potential temperature

\[ \gamma = \frac{c_p}{c_v} \]

\[ \theta_v = \theta (1 + 0.61q_v) \]
Log differentiate eq. of state:

\[ \frac{1}{\gamma} d \ln p = d \ln \rho_m + d \ln \theta_v \]

or, assuming that the differentials of the state variables (A) represent small perturbations from reference state, such that:

\[ \frac{dA}{A} = \frac{A'}{A_0} \ll 1 \]

then

\[ \frac{p'}{\gamma p_o} = \frac{\rho_m'}{\rho_o} + \frac{\theta_v'}{\theta_o} \]

then

\[ \frac{\rho_m'}{\rho_o} = \frac{p'}{\gamma p_o} - \frac{\theta_v'}{\theta_o} \]
Substitute into Eq of Motion:

\[ \rho = \rho_m + \rho_l + \rho_i = \rho_o + \rho'_m + \rho_l + \rho_i \equiv \rho_o \left( 1 + \frac{\rho'_m}{\rho_o} \right) \left( 1 + q_l + q_i \right) \]

then

\[
\frac{\partial u_i}{\partial t} \equiv -u_j \frac{\partial u_i}{\partial x_j} - \epsilon_{i,j,k} f_{j,u_k} - \frac{1}{\rho_o} \frac{\partial p'}{\partial x_i} - \frac{\rho'}{\rho_o} g \delta_{i3}
\]

\[
\frac{\partial u_i}{\partial t} \equiv -u_j \frac{\partial u_i}{\partial x_j} - \epsilon_{i,j,k} f_{j,u_k} - \frac{1}{\rho_o} \frac{\partial p'}{\partial x_i} - \left( \frac{\rho'_m}{\rho_o} + q_l + q_i \right) g \delta_{i3}
\]

Small compared to 1
Define Exner Function

\[ \pi = c_p \left( \frac{p}{p_{oo}} \right)^{R_d / c_p} \]

then

\[ T = \theta \frac{\pi}{c_p} \]

and so:

\[ d \ln \pi = \frac{R_d}{c_p} d \ln p \]

\[ dp = \frac{p / R_d}{\pi / c_p} d\pi = \rho_m \Theta_v d\pi \]
Equation of Motion with Exner Function

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho_m (1 + q_l + q_i)} \frac{\partial p}{\partial x_i} - g \delta_{i3}
\]

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_v \frac{\partial \pi}{\partial x_i} \frac{1}{(1 + q_l + q_i)} - g \delta_{i3}
\]

\[
\frac{\partial u_i}{\partial t} \approx -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{\theta_v}{(1 + q_l + q_i)} \frac{\partial \pi}{\partial x_i} - g \delta_{i3} ; \quad \theta_{vv} = \frac{\theta_v}{(1 + q_l + q_i)}
\]

\[
\frac{\partial u_i}{\partial t} \approx -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_{vv} \frac{\partial \pi}{\partial x_i} - g \delta_{i3}
\]
Equation of Motion with Exner Function (Perturbation Form)

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_j}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{\theta_v}{(1 + q_l + q_l)} \frac{\partial \pi'}{\partial x_i} - \theta'_o \frac{\partial \pi'}{\partial x_i} + \theta'_o \frac{\partial \pi'_o}{\partial x_i} (q_l + q_i) - \frac{\theta'_v}{(1 + q_l + q_i)} \frac{\partial \pi'_o}{\partial x_i} - g \delta_{i3}
\]

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_j}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_{vv} \frac{\partial \pi'}{\partial x_i} + \left( \frac{\theta'_v}{\theta'_o} - q_l - q_i \right) g \delta_{i3}
\]

\[
\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_j}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_{vv} \frac{\partial \pi'}{\partial x_i} + \frac{\theta'_{vv}}{\theta'_o} g \delta_{i3}
\]
Summary of Equations of Motion

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho} \frac{\partial p}{\partial x_i} - g \delta_{i3} & \text{Raw Compressible Form} \\
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} - \frac{\rho'}{\rho} g \delta_{i3} & \text{Perturbation Form 1} \\
\frac{\partial u_i}{\partial t} &\equiv -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho_o} \frac{\partial p'}{\partial x_i} - \frac{\rho'}{\rho_o} g \delta_{i3} & \text{Boussinesq Form 1} \\
\frac{\partial u_i}{\partial t} &\equiv -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \frac{1}{\rho_o} \frac{\partial p'}{\partial x_i} - \frac{p'}{\gamma p_o - \theta_w} \theta_o g \delta_{i3} & \text{Boussinesq Form 2 (TC76)} \\
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_w \frac{\partial \pi}{\partial x_i} - g \delta_{i3} & \text{Exner Raw Form} \\
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_w \frac{\partial \pi'}{\partial x_i} + \frac{\theta_v'}{\theta_o} g \delta_{i3} & \text{Exner Perturbation Form (T92)} \\
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \varepsilon_{i,j,k} f_j u_k - \theta_o \frac{\partial \pi'}{\partial x_i} + \frac{\theta_v'}{\theta_o} g \delta_{i3} & \text{Boussinesq Exner Perturbation Form (KW78)}
\end{align*}
\]
Conservation of Mass

\[
\frac{\partial \rho_m}{\partial t} = -\frac{\partial \rho_m u_j}{\partial x_j} + CN_{lv} + CN_{iv} \quad \text{air + vapor}
\]

\[
\frac{\partial \rho_l}{\partial t} = -\frac{\partial \rho_l (u_j + V_{l,j})}{\partial x_j} - CN_{lv} + CN_{il} \quad \text{liquid water}
\]

\[
\frac{\partial \rho_i}{\partial t} = -\frac{\partial \rho_i (u_j + V_{i,j})}{\partial x_j} - CN_{iv} - CN_{il} \quad \text{ice water}
\]

where \( CN_{ab} \) is the conversion of density from phase "a" to phase "b" and \( V_{a,l} \) is the parcel-relative (terminal) velocity of particle "a" in direction "j"
Equation of State with Exner Function

\[ \frac{1}{p^\gamma} = \frac{\rho_m R_d \theta_v}{p_{oo} \frac{R_d}{c_p}} ; \quad p = p_{oo} \left( \frac{\pi}{c_p} \right)^{c_p/R_d} \]

\[ p_{oo} \left( \frac{\pi}{c_p} \right)^{c_p/R_d} = \frac{\rho_m R_d \theta_v}{p_{oo} \frac{R_d}{c_p}} \]

\[ p_{oo} \left( \frac{\pi}{c_p} \right)^{c_p/R_d} = \rho_m R_d \theta_v \]

\[ \pi^{R_d} = c_p^{R_d} R_d \rho_m \theta_v \]
Log Differentiate

\[ \frac{c_v}{R_d \pi} \frac{d\pi}{\pi} = \frac{d}{\rho_m \theta_v} \]

or

\[ \frac{\partial \pi}{\partial t} = \frac{R_d}{c_v} \frac{\pi}{\rho_m \theta_v} \left[ \theta_v \frac{\partial \rho_m}{\partial t} + \rho_m \frac{\partial \theta_v}{\partial t} \right] \]
Combine with continuity equation

\[ \frac{\partial \pi}{\partial t} = \frac{R_d}{c_v} \frac{\pi}{\rho_m \theta_v} \left[ \theta_v \left( -\frac{\partial \rho_m u_j}{\partial x_j} + CN_{lv} + CN_{iv} \right) + \left( \rho_m \frac{d\theta_v}{dt} - \rho_m u_j \frac{\partial \theta_v}{\partial x_j} \right) \right] \]

\[ \frac{d\pi}{dt} = -\frac{R_d}{c_v} \frac{\pi}{\rho_m \theta_v} \frac{\partial \rho_m \theta_v u_j}{\partial x_j} + \frac{R_d}{c_v} \frac{\pi}{\rho_m \theta_v} \frac{d\theta_v}{dt} + \frac{R_d}{c_v} \frac{\pi}{\rho_m} (CN_{lv} + CN_{iv}) \]

Compressibility and thermal fluxes
Dry diabatic effects
Internal phase conversions
Use the definition of sound speed

The sound speed is:

\[ c_s = \sqrt{\gamma R_d T_v} \]

Hence, alternatively we can write,

\[
\frac{d\pi}{dt} = - \frac{c_s^2}{c_p \rho_m \theta_v^2} \frac{\partial \rho_m \theta_v u_j}{\partial x_j} + \frac{c_s^2}{c_p \rho_m \theta_v^2} \frac{d\theta_v}{dt} + \frac{c_s^2}{c_p \rho_m \theta_v} (CN_{lv} + CN_{iv})
\]
Scale Analysis of Continuity Equation

For scale analysis, we assume a wave solution for all variables:

$$\phi(x_1, x_2, x_3, t) = \hat{\phi}(k_1, k_2, x_3, \omega)e^{i(k_1x_1+k_2x_2+\omega t)}$$

where the horizontal wave number is:

$$k = (k_1 + k_2)^{1/2}$$

and the horizontal wave length is

$$L = \frac{2\pi}{|k|}$$

The order of magnitude of \( \phi \) is given by \( \left| \hat{\phi} \right|_M \). The magnitude of the derivative of \( \phi \) is:

$$L_\phi \left| \frac{\partial \hat{\phi}}{\partial x_3} \right|_M = \left| \hat{\phi} \right|_M \quad \text{or} \quad \frac{1}{L_\phi} = \frac{1}{\left| \hat{\phi} \right|_M} \left| \frac{\partial \hat{\phi}}{\partial x_3} \right|_M$$
Scale Analysis of Continuity Equation

Substitute wave solutions:

\[ u_3 = \hat{u}_3 e^{i(k_1 x_1 + k_2 x_2 + \omega t)} \]
\[ p' = \hat{p}' e^{i(k_1 x_1 + k_2 x_2 + \omega t)} \]
\[ \rho_m' = \hat{\rho}_m' e^{i(k_1 x_1 + k_2 x_2 + \omega t)} \]

into Boussinesq form of vertical equation of motion (linearize and ignore coriolis for this analysis):

\[ \frac{\partial u_3}{\partial t} \equiv -\frac{1}{\rho_o} \frac{\partial p'}{\partial x_3} - \frac{\rho_m'}{\rho_o} g \]

Then:

\[ \frac{\partial u_3}{\partial t} = \hat{u}_3 i \omega = -\frac{1}{\rho_o} \frac{\partial \hat{p}'}{\partial x_3} - \frac{\hat{\rho}_m'}{\rho_o} g \]

For convection, buoyancy an pressure gradient play important roles, thus:

\[ g \left( \frac{\hat{\rho}_m'}{\rho_o} \right)_M = \left| \frac{1}{\rho_o} \frac{\partial \hat{p}'}{\partial x_3} \right|_M \equiv |\omega \hat{u}_3|_M \]
Define Density Scale Height

Define Density Scale Height:

\[ \frac{1}{H_{\rho}} = \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial x_3} \]

then
Scale Analysis of Continuity Equation

Consider the Boussinesq form of the continuity equation, neglecting condensation:

\[
\frac{\partial \rho_m}{\partial t} = - \frac{\partial \rho_m u_j}{\partial x_j}
\]

Boussinesq approximation:

\[
\frac{\partial \rho_m^*}{\partial t} = - \frac{\partial (\rho_o + \rho^*) u_1}{\partial x_1} - \frac{\partial (\rho_o + \rho^*) u_2}{\partial x_2} - \frac{\partial (\rho_o + \rho^*) u_3}{\partial x_3}
\]

or

\[
\frac{\partial \rho_m^*}{\partial t} = -u_1 \frac{\partial \rho_m^*}{\partial x_1} - u_2 \frac{\partial \rho_m^*}{\partial x_2} - u_3 \frac{\partial \rho_m^*}{\partial x_3} - \rho_o \frac{\partial \rho_o}{\partial x_1} - \rho_o \frac{\partial \rho_o}{\partial x_2} - \rho_o \frac{\partial \rho_o}{\partial x_3}
\]

\[
-\rho_o \left(1 + \frac{\rho_o^*}{\rho_o}\right) \frac{\partial u_1}{\partial x_1} - \rho_o \left(1 + \frac{\rho_o^*}{\rho_o}\right) \frac{\partial u_2}{\partial x_2} - \rho_o \left(1 + \frac{\rho_o^*}{\rho_o}\right) \frac{\partial u_3}{\partial x_3}
\]
Scale Analysis of Continuity Equation

\[
\frac{\partial \rho_m'}{\partial t} = -u_j \frac{\partial \rho_o'}{\partial x_{j(j=1,2)}} - u_3 \frac{\partial \rho_m'}{\partial x_3} - u_3 \frac{\partial \rho_o}{\partial x_3} - \rho_o \frac{\partial u_j}{\partial x_{j(j=1,2)}} - \rho_o \frac{\partial u_3}{\partial x_3}
\]
Scale

Divide (a) by (d):

\[
\begin{align*}
\left| \frac{\omega \hat{\rho}'}{M} \right| &= \left| \frac{\omega \hat{\rho}'}{H_\rho} \right| = \frac{H_\rho \omega^2}{\omega \hat{u}_3} \left| \frac{\hat{\rho}'}{\rho_o} \right| \\
\hat{u}_3 \frac{\partial \rho_o}{\partial x_3} |_M &= \hat{u}_3 \frac{\rho_o}{H_\rho} |_M
\end{align*}
\]

Divide (b) by (d):

\[
\begin{align*}
\left| \frac{\nu H k_j \hat{\rho}'}{M} \right| &= \left| \frac{\nu H 2\pi H_\rho \hat{\rho}'}{L_i} \right| = \frac{\nu H}{\hat{u}_3} \left| \frac{2\pi H_\rho}{L_i} \hat{\rho}'}{\rho_o} \right| \\
\hat{u}_j \frac{\partial \hat{\rho}'}{\partial x_j} |_M &= \hat{u}_3 \frac{\rho_o}{H_\rho} |_M
\end{align*}
\]
Scale

Divide (c) by (d):

\[
\frac{\hat{u}_3 \frac{\partial \hat{\rho}'}{\partial x_3}}{M} = \frac{\hat{u}_3 \hat{\rho}'}{H'} = \frac{H_{\rho}}{H'} \frac{\hat{\rho}'}{\rho_o}
\]

Divide (e) by (d):

\[
\frac{\rho_o \frac{\partial u_j}{\partial x_j}}{M} = \frac{\rho_o \nu_H k_j}{M} = \frac{2\pi}{L_j} \frac{H_{\rho}}{H'} \frac{\nu_H}{\hat{u}_3}
\]
Scale

Divide (f) by (d)

\[ \frac{\rho_o}{\hat{u}_3 \frac{\partial \rho_o}{\partial x_3}}_M \frac{\hat{u}_3 \hat{\rho}_o}{L_{u_3}}_M = \frac{H_{\rho}}{L_{u_3}} \]
Evaluation of Analysis

• Restrict system to low frequencies, removing sound waves, ie:

\[ |\omega^2| \ll \frac{g}{H \rho} \]

• Then:
  – Term (a) is negligible
  – Generally,

\[ \left| \frac{\mathcal{V}_h}{\hat{u}_3} \right| \leq \frac{L_i}{2\pi H \rho} \]

So, since

\[ \left| \frac{\hat{\rho}'}{\rho_o} \right| \ll 1 \]

Term (b) is negligible
• For deep convection,

\[ H_\rho \sim H'_\rho. \]

• And so using Boussinesq approximation again, (c) is negligible.

• Also, for deep convection,

\[ L_{u_3} \sim H_\rho \text{ and } L_{u_3} \sim L_i. \]

• Showing that (d) and (e) are important
Hence

$$\frac{\partial \rho_u}{\partial x_j} = 0$$

- Anelastic Continuity Equation.
\[ \frac{\partial u_j}{\partial x_j} = 0 \]

\[ L_{u_3} \ll H_\rho \]

- Incompressible of shallow form of the Continuity Equation.