

Equations Satisfied by a Moving Fluid

4.1 Properties of a Material Element

When a fluid is in motion, its properties are functions both of spatial position

$$\mathbf{x} \equiv (x, y, z) \quad (4.1.1)$$

and time t . In other words, for any property γ ,

$$\gamma = \gamma(x, y, z, t) \equiv \gamma(\mathbf{x}, t). \quad (4.1.2)$$

(The symbol \equiv , meaning "is identical to," is used here to relate different ways of writing the same expression, e.g., in scalar notation on one side and vector notation on the other side.) Now the concepts of the state of a fluid apply to a particular sample (or "parcel") that will move around when the fluid is in motion. Since nearby particles of fluid may move apart in time, it is necessary to think of an infinitesimally small sample that will retain its identity. This will be called a material element of fluid (Batchelor, 1967, Chapter 2).

Suppose now that this material element has position \mathbf{x} at time t given by

$$\mathbf{x} = \mathbf{x}(t). \quad (4.1.3)$$

Then the property γ for this material element will vary with time according to

$$\gamma = \gamma(x(t), y(t), z(t), t) \equiv \gamma(\mathbf{x}(t), t). \quad (4.1.4)$$

It follows that the rate of change of γ for the material element is given by

$$\frac{d\gamma}{dt} = \frac{\partial\gamma}{\partial t} + \frac{\partial\gamma}{\partial x} \frac{dx}{dt} + \frac{\partial\gamma}{\partial y} \frac{dy}{dt} + \frac{\partial\gamma}{\partial z} \frac{dz}{dt} \equiv \frac{\partial\gamma}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\gamma. \quad (4.1.5)$$

Now $d\mathbf{x}/dt$ is the rate of change of position of a material element, i.e., the fluid velocity

$$d\mathbf{x}/dt = \mathbf{u} \equiv (u, v, w). \quad (4.1.6)$$

Therefore, for a material element, $d\gamma/dt$ is equal to $D\gamma/Dt$, where $D\gamma/Dt$ is defined by

$$\frac{D\gamma}{Dt} \equiv \frac{\partial\gamma}{\partial t} + u \frac{\partial\gamma}{\partial x} + v \frac{\partial\gamma}{\partial y} + w \frac{\partial\gamma}{\partial z} \equiv \frac{\partial\gamma}{\partial t} + \mathbf{u} \cdot \nabla\gamma. \quad (4.1.7)$$

Note that the symbol D/Dt is defined by (4.1.7) and so has this meaning irrespective of the context. The symbol d/dt , on the other hand, means the time derivative of a quantity that is a function of time only. (Despite this fact, the symbol d/dt is used in some textbooks to have the same meaning as that given to D/Dt here. This does not often lead to confusion, but it is better to have different symbols for operators that have different meanings.)

The usefulness of the operator D/Dt can be illustrated immediately by considering the "concentration" equations for air and seawater. If molecular diffusion can be ignored, the material element will always consist of the same particles and so the mass of each constituent will remain constant. Since the salinity s is the mass of dissolved salt per unit mass of fluid, s will also remain constant, and so

$$Ds/Dt = 0. \quad (4.1.8)$$

Similarly, for the atmosphere, the specific humidity q is the mass of water vapor per unit mass of air. Thus if no phase changes are taking place,

$$Dq/Dt = 0. \quad (4.1.9)$$

A similar equation holds for any quantity that is conserved by material elements.

4.2 Mass Conservation Equation

As a material element moves, its mass remains constant but its volume may alter. Therefore, its density may change, but in a way that is dependent on the field of motion. The equation relating the rate of change of density to the field of motion is called the mass conservation equation. There are two equivalent ways of writing this equation, corresponding to different methods of derivation. The first method, which was used by Euler (1755) in his paper on the equations of motion, considers the changes following a material element. The second method considers the changes for a fixed volume element. These two different approaches may be applied to the other equations of motion as well, and both will be considered here.

The first method requires calculation of the fractional rate of change $v_s^{-1} Dv_s/Dt$ of the specific volume v_s of a material element. Here this will be calculated for an infinitesimal Cartesian element as shown in Fig. 4.1 [for a more general discussion, see Batchelor (1967, Sections 2.2 and 3.1)]. Consider an element that (Fig. 4.1a) is initially rectangular, with sides δx , δy , and δz . A short time later (Fig. 4.1b), the element will be slightly distorted. To first order in δx , δy , and δz , the volume changes only

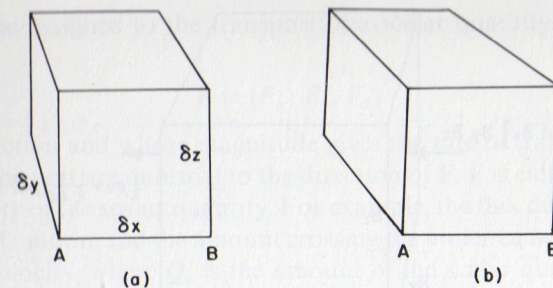


Fig. 4.1. A material volume element that is initially rectangular (a) with sides δx , δy , δz will a short time later (b) be displaced and slightly distorted. To first order in δx , δy , and δz , the volume change is due only to changes in lengths of the sides. Changes in angles between the sides do not alter the volume to this order.

because of small changes in the lengths of the sides, slight rotations of the edges not being significant to this order. Therefore the fractional rate of change of volume is

$$\frac{1}{\delta x \delta y \delta z} \frac{D}{Dt} (\delta x \delta y \delta z) = \frac{1}{\delta x} \frac{D}{Dt} (\delta x) + \frac{1}{\delta y} \frac{D}{Dt} (\delta y) + \frac{1}{\delta z} \frac{D}{Dt} (\delta z). \quad (4.2.1)$$

But the first term

$$\frac{1}{\delta x} \frac{D}{Dt} (\delta x) = \frac{1}{\delta x} \frac{D}{Dt} (x(B) - x(A)) = \frac{1}{\delta x} (u(B) - u(A)) \rightarrow \frac{\partial u}{\partial x} \quad \text{as } \delta x \rightarrow 0,$$

and similarly for the other terms. It follows that the fractional rate of change of specific volume is equal to the divergence $\nabla \cdot \mathbf{u}$ of the velocity, i.e.,

$$\frac{1}{v_s} \frac{Dv_s}{Dt} = \nabla \cdot \mathbf{u} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \quad (4.2.2)$$

For fluid mechanics problems it is usually more convenient to use density ρ as a variable rather than its reciprocal v_s . Then (4.2.2) takes the form

$$\rho^{-1} D\rho/Dt + \nabla \cdot \mathbf{u} = 0. \quad (4.2.3)$$

The mass conservation equation (4.2.3) is fundamental in all problems involving fluid motion. An alternative form of it, which will be derived from first principles, can be obtained from (4.2.3) using (4.1.7), the definition of D/Dt , and (4.2.2), the definition of the divergence operator. First, (4.2.3) gives

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0,$$

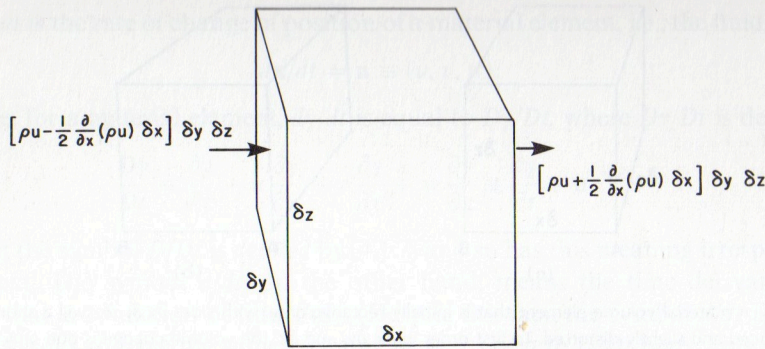
i.e.,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0,$$

or

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (4.2.4)$$

The second method of derivation considers the mass balance for a small volume element fixed in space (Fig. 4.2) and leads directly to the form (4.2.4). For such an



a direction can be assigned to the transport of a scalar quantity, so one can define a vector

$$\mathbf{F} \equiv (F_x, F_y, F_z)$$

that has this direction and whose magnitude gives the rate of transport of the scalar quantity across the unit area normal to the direction of \mathbf{F} . \mathbf{F} is called the flux per unit area or *flux density* of the scalar quantity. For example, the flux due to fluid motion is in the direction of motion, and the amount crossing the unit area normal to the motion is Q_v times the velocity, where Q_v is the amount of the scalar quantity contained in the unit volume. In this case,

$$\mathbf{F} = Q_v \mathbf{u}.$$

The flux (or rate of transport) through an element of area δA , which is not at right angles to \mathbf{F} , can be calculated by simple geometry. Figure 4.3 shows a cut through the element of area, the plane of the drawing being the one that contains both \mathbf{F} and the normal to δA . $\delta S = \delta A \cos \alpha$ is the projection of δA on the plane normal to \mathbf{F} and α is the angle between the planes of δA and δS . The flux is equal to

$$F \delta S = F \delta A \cos \alpha = F \cos \alpha \delta A,$$

where F is the magnitude of \mathbf{F} , and so the flux per unit area is $F \cos \alpha$, i.e., the component of \mathbf{F} normal to the area concerned.

For the volume element depicted in Fig. 4.2 the flux across the side of area $\delta y \delta z$ is $F_x \delta y \delta z$, where F_x is the x-component of the flux. Following the same argument as for mass, the difference in flux between the two sides of area $\delta y \delta z$ is

$$\partial F_x / \partial x \cdot \delta x \delta y \delta z$$

and the rate of loss of the scalar quantity per unit volume is

$$\nabla \cdot \mathbf{F} \equiv \partial F_x / \partial x + \partial F_y / \partial y + \partial F_z / \partial z.$$

In other words, the equation satisfied by Q_v , the quantity per unit volume, is

$$\partial Q_v / \partial t + \nabla \cdot \mathbf{F} = 0. \quad (4.3.1)$$

The mass conservation equation (4.2.4) is the special case in which $Q_v = \rho$ and $\mathbf{F} = \rho \mathbf{u}$. The equation for salinity or humidity is another special case, in which $Q_v = \rho s$ is the mass of salt (or water vapor) per unit volume. The advective flux (i.e., the flux due to fluid motion) is $\rho s \mathbf{u}$, so if there is no other means of transporting salt

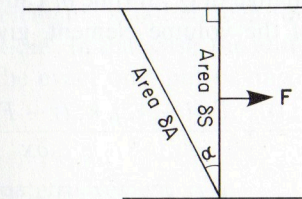


Fig. 4.3. A diagram for calculating the flux across an area element δA when the flux density is \mathbf{F} . The sketch shows a section through the area element in the plane that contains \mathbf{F} and the normal to the area element. Lines are drawn through the boundary of δA parallel to \mathbf{F} , and δS is the projection of δA on a plane normal to \mathbf{F} .

(or water vapor), (4.3.1) becomes

$$\partial(\rho s)/\partial t + \nabla \cdot (\rho s \mathbf{u}) = 0. \quad (4.3.2)$$

This equation could also be derived from (4.1.8) and (4.2.3), so it is merely an alternative way of expressing the salinity (or water vapor) balance.

However, there is another means of transporting salt (or water vapor), molecular diffusion, which occurs when there are salinity (or humidity) gradients. This is a very slow process and is therefore neglected in most problems considered in this book. The diffusive flux is in the opposite direction to ∇s , the gradient of s , i.e., it carries salt from regions of high concentration to regions of low concentration, and is equal to (Batchelor, 1967, Section 1.6)

$$-\rho \kappa_D \nabla s,$$

where κ_D , the *diffusivity* of salt in water, is a coefficient that determines the rate of diffusion and can depend on the state of the fluid, i.e., on the temperature, pressure, and salinity. Values of κ_D for common substances can be found in Weast (1971–1972), e.g., the value for salt in water is $1.5 \times 10^{-9} \text{ m}^2 \text{ s}^{-1}$ at 25°C , and that for water vapor in air is $2.4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ at 8°C [see also List (1951, Table 113)]. When the diffusive flux is included,

$$\mathbf{F} = \rho s \mathbf{u} - \rho \kappa_D \nabla s \quad (4.3.3)$$

and (4.3.1) becomes

$$\partial(\rho s)/\partial t + \nabla \cdot (\rho s \mathbf{u} - \rho \kappa_D \nabla s) = 0. \quad (4.3.4)$$

[Note: Salt diffusion can also be caused by temperature and pressure gradients (Fofonoff, 1962), but the effects are minor on the very small scales for which diffusion is important.]

4.3.1 Finite-Difference Formulation for Numerical Models

The concepts of fluxes across the sides of volume elements are also used in *numerical models* of the atmosphere and ocean [see, e.g., Haltiner (1971), Bryan (1969), Mesinger and Arakawa (1976)]. Such models may be divided into a set of volume elements like that shown in Fig. 4.2, except that now the dimensions are finite. Each volume element is identified by integers (i, j, k) that determine its position in a grid (see Fig. 4.4) and the value $Q_v(i, j, k; t)$ denotes the average value of Q_v over the volume element (i, j, k) at time t . The change in Q_v in time δt can be calculated from the sum of the fluxes over the sides of the volume element, giving as the finite-difference analog of (4.3.1),

$$\begin{aligned} \frac{Q_v(i, j, k; t + \delta t) - Q_v(i, j, k; t)}{\delta t} + \frac{F_x(i + \frac{1}{2}, j, k; \bar{t}) - F_x(i - \frac{1}{2}, j, k; \bar{t})}{\delta x} \\ + \frac{F_y(i, j + \frac{1}{2}, k; \bar{t}) - F_y(i, j - \frac{1}{2}, k; \bar{t})}{\delta y} \\ + \frac{F_z(i, j, k + \frac{1}{2}; \bar{t}) - F_z(i, j, k - \frac{1}{2}; \bar{t})}{\delta z} = 0, \quad (4.3.5) \end{aligned}$$

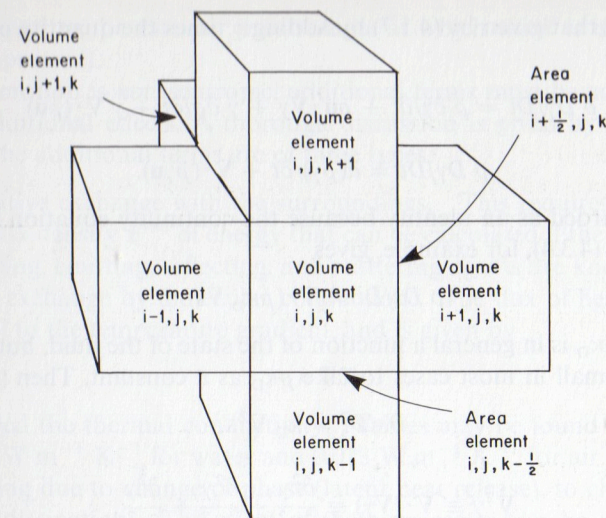


Fig. 4.4. A typical arrangement of volume elements (of finite size) in a numerical model. Volume elements, and quantities associated with them, are identified by integers as shown. To identify area elements, one of the integers is replaced by a value halfway between the integer values associated with the two volume elements that it separates.

where δx , δy , and δz are the sides of the volume element and $F_x(i + \frac{1}{2}, j, k; \bar{t})$ is the average x -component of the flux per unit area across face $(i + \frac{1}{2}, j, k)$ during the time interval from t to $t + \delta t$. Face $(i + \frac{1}{2}, j, k)$ is the one common to volume elements (i, j, k) and $(i + 1, j, k)$.

With the interpretation given above, (4.3.5) is exact. The approximation comes in when the components of \mathbf{F} are calculated in terms of other quantities. For instance, in the *mass conservation* equation, where $Q_v = \rho$, $F_x(i + \frac{1}{2}, j, k; \bar{t})$ is the average value of ρu across the face $(i + \frac{1}{2}, j, k)$. However, changes in ρu are calculated from the *momentum* equations, where ρu is interpreted as the average value of the x -component of momentum per unit volume over a *volume* element. Some form of approximation is required to relate the average value over a face to average values over appropriate volume elements. This approximation must have the property that if the size of the volume elements tends toward zero, the approximation becomes more and more accurate. This should be true of all finite-difference schemes, whether or not the formulas are readily interpreted in the way described above. Unfortunately, it is not always possible to choose elements small enough for the finite-difference solutions to be close to the exact solutions. In that case, the numerical model is best interpreted as a system distinct from the exact one, but having, it is hoped, closely analogous behavior.

4.3.2 Changes Following a Material Element

Often it is desirable to modify an equation like (4.3.4) to give changes following a material element rather than changes at a fixed position. This can be done by using, for any variable γ (usually a quantity per unit mass), an expression for $\rho D\gamma/Dt$ that

is modified from that given by (4.1.7) by adding γ times the quantity equated to zero in (4.2.4). Thus

$$\rho D\gamma/Dt = \rho \partial\gamma/\partial t + \rho \mathbf{u} \cdot \nabla \gamma + \gamma \partial\rho/\partial t + \gamma \nabla \cdot (\rho \mathbf{u})$$

i.e.,

$$\rho D\gamma/Dt = \partial(\rho\gamma)/\partial t + \nabla \cdot (\rho\gamma \mathbf{u}). \quad (4.3.6)$$

This can be regarded as an identity because the continuity equation (4.2.4) is exact. Applying this to (4.3.4), for example, gives

$$\rho Ds/Dt = \nabla \cdot (\rho \kappa_D \nabla s). \quad (4.3.7)$$

The coefficient $\rho \kappa_D$ is in general a function of the state of the fluid, but the variations are sufficiently small in most cases to take $\rho \kappa_D$ as a constant. Then (4.3.7) becomes

$$Ds/Dt = \kappa_D \nabla^2 s, \quad (4.3.8)$$

where

$$\nabla^2 \gamma \equiv \nabla \cdot (\nabla \gamma) \equiv \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} + \frac{\partial^2 \gamma}{\partial z^2}. \quad (4.3.9)$$

4.4 The Internal Energy (or Heat) Equation

This equation has a simple form for fluid elements that do not exchange heat with their surroundings and that retain a fixed composition. In such cases, the motion is said to be *isentropic*, i.e., the entropy of a material element is fixed, and the state of such an element does not change during the motion. Therefore the relationships among state variables given in Chapter 3 apply to the element at all times, and the form of equation depends on which variables are used to describe the state of the fluid. In terms of specific entropy η or potential temperature θ ,

$$D\eta/Dt \equiv c_p(p_r, \theta) \theta^{-1} D\theta/Dt = 0 \quad (4.4.1)$$

by (3.7.6), where c_p is the specific heat and p_r is the reference pressure. Alternative forms follow from (3.2.1), (3.2.6), and (3.6.1), namely,

$$T \frac{D\eta}{Dt} \equiv \frac{DE}{Dt} + p \frac{Dv_s}{Dt} \equiv c_p \frac{DT}{Dt} - \frac{\alpha T}{\rho} \frac{Dp}{Dt} = 0, \quad (4.4.2)$$

where T is temperature, E internal energy per unit mass, v_s specific volume, and α the thermal expansion coefficient. All the above equations can be expressed as balances for a fixed volume element by using the general relationship (4.3.6), and diagrams like Fig. 4.2 can be drawn to visualize the balances that occur. For instance, the equation in terms of the internal energy E may be written

$$\rho DE/Dt \equiv \partial(\rho E)/\partial t + \nabla \cdot (\rho E \mathbf{u}) \equiv -p v_s^{-1} Dv_s/Dt = -p \nabla \cdot \mathbf{u}, \quad (4.4.3)$$

where (4.2.2) has been used to give Dv_s/Dt . The physical interpretation of (4.4.3) is that the internal energy in a fixed volume can change by advection across the sides (the term $\rho E \mathbf{u}$) or by compression or expansion of the fluid in the volume (the right-

hand side is equal to the rate per unit volume at which work is done on the fluid when it is compressed).

When the motion is *not* isentropic, additional terms must be added to (4.4.3) to include the additional effects. A thorough discussion is given by Batchelor (1967, Section 3.4). The additional terms are of three types:

(i) Radiative exchange with the surroundings. This requires a knowledge of the radiative flux density \mathbf{F}^{rad} of energy that can be calculated if the distribution and state of absorbing, emitting, reflecting, and scattering agents are known.

(ii) Heat exchange by molecular conduction. The flux of heat by this means is proportional to the temperature gradient, and is given by

$$-k \nabla T,$$

where k is called the thermal conductivity. [Values may be found in Weast (1971, 1972), e.g., $0.6 \text{ W m}^{-1} \text{ K}^{-1}$ for water and $0.023 \text{ W m}^{-1} \text{ K}^{-1}$ for air.]

(iii) Heating due to change of phase (latent heat release), to chemical reaction or to viscous dissipation. The effect of these processes can be represented by a term Q_H , which gives the rate of heating per unit volume. The modified form of (4.4.3) is then

$$\partial(\rho E)/\partial t + \nabla \cdot (\rho E \mathbf{u} + \mathbf{F}^{\text{rad}} - k \nabla T) = Q_H - p \nabla \cdot \mathbf{u}, \quad (4.4.4)$$

where the quantity

$$\mathbf{F} = \rho E \mathbf{u} + \mathbf{F}^{\text{rad}} - k \nabla T \quad (4.4.5)$$

that appears on the left-hand side may be called the heat flux density.

Alternative forms of this equation can be obtained by using the relations among state variables [summarized in (4.4.1) and (4.4.2)] and the expression (4.3.6) relating point derivatives to derivatives following the motion. With temperature as state variable, the equation becomes

$$\rho c_p DT/Dt - \alpha T Dp/Dt = \nabla \cdot (k \nabla T - \mathbf{F}^{\text{rad}}) + Q_H. \quad (4.4.6)$$

Alternatively, in terms of potential temperature

$$\rho T c_p(p_r, \theta) \theta^{-1} D\theta/Dt = \nabla \cdot (k \nabla T - \mathbf{F}^{\text{rad}}) + Q_H. \quad (4.4.7)$$

Note that if k is a constant, it may be taken outside the bracket in the above expressions, giving rise to the combination

$$\kappa = k/\rho c_p, \quad (4.4.8)$$

where κ is called the thermal diffusivity. Typical values are $1.4 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}$ for water and $2 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ for air. These values are so small that thermal conduction is not of direct importance for the scales mainly considered in this book, and hence is usually neglected. The radiative term may be quite important in the atmosphere, but not in the ocean except for the top 30 m or so. The internal heating term Q_H is rarely important except in those parts of the atmosphere where latent heat release is taking place due to condensation. Assuming the latent heat release to be pseudoadiabatic (see Section 3.8), Q_H is nonzero only when (a) q has attained the saturation value q_s

When the point \mathbf{x}_r is the position of a material volume element, the derivative d/dt is the same as D/Dt (see Section 4.1), so (4.5.1) becomes

$$D\mathbf{u}/Dt + 2\boldsymbol{\Omega} \times \mathbf{u} = -\rho^{-1} \nabla p - \nabla\Phi, \quad (4.5.5)$$

where the subscript r will be implied rather than written explicitly from this point onward. Φ is the geopotential defined by

$$\Phi = \Phi_v - \frac{1}{2}\Omega^2 x_r^2, \quad (4.5.6)$$

i.e., is the sum of the gravitational potential Φ_v and centrifugal potential $-\frac{1}{2}\Omega^2 x_r^2$ as defined in Section 3.5. Note also that (4.5.5) is not altered by a change in origin of the axes, so it is not necessary to have the origin on the axis of rotation when using (4.5.5).

4.5.2 Momentum Balance for a Fixed Volume Element

Multiplication of (4.5.5) by ρ and use of the identity (4.3.6) for each component in turn give an alternative form of the equation of motion, namely,

$$\partial(\rho u)/\partial t + \nabla \cdot (\rho u \mathbf{u}) + 2\Omega_y \rho w - 2\Omega_z \rho v = -\partial p/\partial x, \quad (4.5.7)$$

$$\partial(\rho v)/\partial t + \nabla \cdot (\rho v \mathbf{u}) + 2\Omega_z \rho u - 2\Omega_x \rho w = -\partial p/\partial y, \quad (4.5.8)$$

$$\partial(\rho w)/\partial t + \nabla \cdot (\rho w \mathbf{u}) + 2\Omega_x \rho v - 2\Omega_y \rho u = -\partial p/\partial z - \rho g. \quad (4.5.9)$$

The axes (at the element concerned) have been chosen so that the z axis points vertically upward, i.e., in the direction of

$$\mathbf{g} = \nabla\Phi, \quad (4.5.10)$$

where $-\mathbf{g}$ is the acceleration due to gravity of magnitude g [cf. (3.5.2)]. The angular velocity $\boldsymbol{\Omega}$ has been written in terms of its components $(\Omega_x, \Omega_y, \Omega_z)$. These equations can be interpreted in terms of the momentum balance (per unit volume) for a fixed volume element (such as that shown in Fig. 4.2). The rate of change of momentum (first term) is determined by the flux of momentum across the sides of the element (second term), the Coriolis force acting on the element (last two terms on the left-hand side), the net force resulting from the pressure on the sides (first term on the right-hand side) and the gravitational force [last term in (4.5.9)].

4.5.3 Effects of Viscosity

Although viscosity is not of direct importance for the scales of motion considered in this book, it is of indirect importance as a means of removing mechanical energy from the system. Viscosity gives rise to stresses on the surface of a material volume element that may be related to the rate of strain. A detailed discussion is given by Batchelor (1967, Section 3.3). It turns out that if, on the scales for which viscosity is important, viscosity changes and compressibility effects can be ignored, then the

effect on momentum is the same as that of a diffusion process (see Section 4.3) and can be taken into account by adding diffusive fluxes to (4.5.7)–(4.5.9), i.e.,

$$\partial(\rho u)/\partial t + \nabla \cdot (\rho u \mathbf{u} - \mu \nabla u) + 2\Omega_y \rho w - 2\Omega_z \rho v = -\partial p/\partial x, \quad (4.5.11)$$

$$\partial(\rho v)/\partial t + \nabla \cdot (\rho v \mathbf{u} - \mu \nabla v) + 2\Omega_z \rho u - 2\Omega_x \rho w = -\partial p/\partial y, \quad (4.5.12)$$

$$\partial(\rho w)/\partial t + \nabla \cdot (\rho w \mathbf{u} - \mu \nabla w) + 2\Omega_x \rho v - 2\Omega_y \rho u = -\partial p/\partial z - \rho g, \quad (4.5.13)$$

where μ is called the *viscosity* of the fluid. [Values of μ are tabulated by Weast (1971–1972), e.g., $10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$ for water and $1.7 \times 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1}$ for air.] The value of μ depends on the state of the fluid, but the variations are sufficiently slow in most cases to take μ as a constant. Then the modified form of (4.5.5) is

$$D\mathbf{u}/Dt + 2\boldsymbol{\Omega} \times \mathbf{u} = -\rho^{-1} \nabla p - \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (4.5.14)$$

where

$$\nu = \mu/\rho \quad (4.5.15)$$

is called the *kinematic viscosity* [which has a value of $10^{-6} \text{ m}^2 \text{ s}^{-1}$ for water and $1.4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ for air at 1000 mbar—see List (1951, Table 113)], and $\nabla^2 \mathbf{u}$ is the vector with components

$$\nabla^2 \mathbf{u} = (\nabla^2 u, \nabla^2 v, \nabla^2 w). \quad (4.5.16)$$

4.5.4 Perturbation Pressure and Perturbation Density

For large-scale motions in the ocean and atmosphere, the dominant terms by far in the equation of motion (4.5.14) are the gravitational acceleration \mathbf{g} and the vertical component of the pressure gradient, which approximately balances it. In other words, none of the other acceleration terms in (4.5.14) approaches the gravitational acceleration. In the atmosphere, for instance, winds are of order 10 m s^{-1} , so the Coriolis acceleration is about 10^{-3} m s^{-2} , i.e., less than the gravitational acceleration by a factor of 10,000!

Hence it is desirable to define a perturbation pressure and a perturbation density as departures from an equilibrium solution

$$p = p_0(z), \quad \rho = \rho_0(z), \quad (4.5.17)$$

of the type considered in Section 3.5, i.e., which satisfied the hydrostatic equation

$$dp_0/dz = -g\rho_0. \quad (4.5.18)$$

The perturbation pressure p' and perturbation density ρ' are defined by

$$p = p_0(z) + p', \quad \rho = \rho_0(z) + \rho', \quad (4.5.19)$$

in which case (4.5.14) becomes

$$\rho(D\mathbf{u}/Dt + 2\boldsymbol{\Omega} \times \mathbf{u}) = -\nabla p' - \rho' \mathbf{g} + \nabla \cdot (\mu \nabla \mathbf{u}). \quad (4.5.20)$$

In the special case of a homogeneous fluid, i.e., one of uniform density, ρ' is zero. Otherwise the term $-\rho' \mathbf{g}$ represents a force per unit volume called the *buoyancy* force

since an element with negative ρ' is relatively buoyant and therefore experiences an upward force due to the action of gravity.

4.6 Mechanical Energy Equation

The set of equations governing the behavior of the ocean and atmosphere has now been derived. They are summarized in Section 4.10 and consist of (a) the mass conservation equation, (b) the equation of motion, (c) the internal energy or heat equation, (d) the equation of state, and (e) the equations for the concentrations of constituents such as salt and water vapor. From this set, other useful equations can be derived, using elementary calculus. In this section, the equation for *mechanical* or *kinetic* energy is considered.

The kinetic energy per unit mass is defined as $\frac{1}{2}\mathbf{u}^2$.^{*} An equation for the rate of change of this quantity following a material volume element is obtained by taking the scalar product of (4.5.20) with \mathbf{u} , which gives

$$\rho D(\frac{1}{2}\mathbf{u}^2)/Dt = -wg\rho' + \nabla \cdot (-p'\mathbf{u} + \mu \nabla(\frac{1}{2}\mathbf{u}^2)) - \rho\epsilon + p' \nabla \cdot \mathbf{u}, \quad (4.6.1)$$

where

$$\epsilon = \nu \left(\left(\frac{\partial \mathbf{u}}{\partial x} \right)^2 + \left(\frac{\partial \mathbf{u}}{\partial y} \right)^2 + \left(\frac{\partial \mathbf{u}}{\partial z} \right)^2 \right) \quad (4.6.2)$$

is always positive and is called the *dissipation* rate (see below). Another version of (4.6.1) has the primes removed and is obtained from (4.5.14) by the same procedure. Note that the scalar product of \mathbf{u} with the Coriolis acceleration in (4.5.20) is identically zero, so there is no Coriolis term in (4.6.1).

Equation (4.6.1) can be converted to one for a fixed volume element by applying (4.3.6), which gives

$$\partial(\frac{1}{2}\rho\mathbf{u}^2)/\partial t + \nabla \cdot \mathbf{F}' = -wg\rho' - \rho\epsilon + p' \nabla \cdot \mathbf{u}, \quad (4.6.3)$$

where

$$\mathbf{F}' = (p' + \frac{1}{2}\rho\mathbf{u}^2)\mathbf{u} - \mu \nabla(\frac{1}{2}\mathbf{u}^2) \quad (4.6.4)$$

will be called the *energy flux density vector* because it gives a rate of flow of energy per unit area. It is not uniquely defined, however. For instance, any vector with zero divergence could be added to \mathbf{F}' without altering (4.6.3). As before, there is an alternative version with the primes removed, i.e., with pressure in place of perturbation pressure and density in place of perturbation density. In this case \mathbf{F} is used in place of \mathbf{F}' in (4.6.3) and (4.6.4).

In the special case of a fluid of uniform density, $\rho' = 0$ and $\nabla \cdot \mathbf{u} = 0$ by (4.2.3), so (4.6.3) simplifies to

$$\partial(\frac{1}{2}\rho\mathbf{u}^2)/\partial t + \nabla \cdot \mathbf{F}' = -\rho\epsilon. \quad (4.6.5)$$

^{*} In order to simplify notation, \mathbf{u}^2 has been written in place of $\mathbf{u} \cdot \mathbf{u}$ throughout.

The physical interpretation of this equation is illustrated in Fig. 4.6 (cf. Fig. 4.2). The mass of the fluid in the volume element shown is $\rho \delta x \delta y \delta z$, so its kinetic energy is $\frac{1}{2}\rho\mathbf{u}^2 \delta x \delta y \delta z$ by definition. This can change (a) by energy transfer across the sides of the element or (b) by energy loss within the element. The rate of transfer of energy due to a flux \mathbf{F}' is shown for two of the sides in the figure, F'_x being the component of \mathbf{F}' in the direction of the x axis. Adding the contributions from all sides gives a net gain of energy of

$$-\left(\frac{\partial}{\partial x} F'_x + \frac{\partial}{\partial y} F'_y + \frac{\partial}{\partial z} F'_z \right) \delta x \delta y \delta z \equiv -\nabla \cdot \mathbf{F}' \delta x \delta y \delta z$$

to the appropriate order of approximation. This gives rise to the term $\nabla \cdot \mathbf{F}'$ in (4.6.5) after dividing by the volume $\delta x \delta y \delta z$ and taking the limit as the volume shrinks to zero.

By (4.6.4), the individual contributions to the rate $F'_x \delta y \delta z$ of energy transfer across a face of area $\delta y \delta z$ are

- (i) $p' u \delta y \delta z$,
- (ii) $\frac{1}{2} \rho \mathbf{u}^2 u \delta y \delta z$,
- (iii) $-\mu \partial(\frac{1}{2}\mathbf{u}^2)/\partial x \cdot \delta y \delta z$.

The first contribution is the product of $p' \delta y \delta z$, the normal force on the face of the element due to the pressure perturbation p' , and u , the rate of movement in the direction of the force. It is therefore the rate of working by the pressure force on that side. The second contribution is the rate of advection of kinetic energy across the face. The third contribution can be interpreted as the rate of diffusion of kinetic energy across the face due to viscous processes.

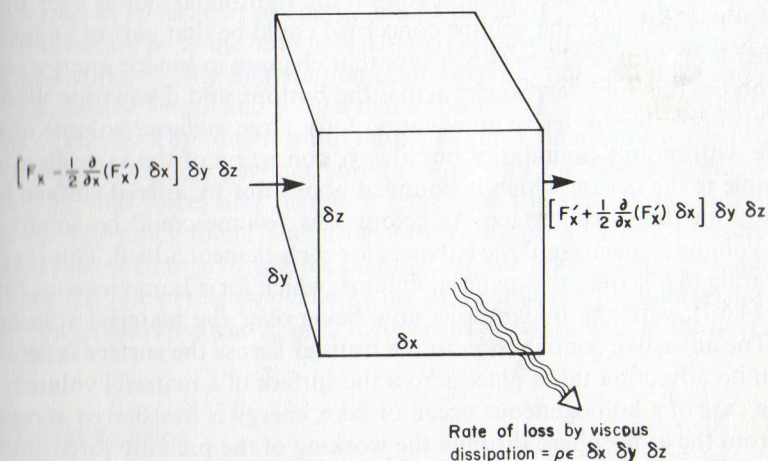


Fig. 4.6. The mechanical energy balance for a fixed rectangular volume element in a homogeneous fluid of density ρ . Fluxes across one pair of faces are shown, where F'_x is the x -component of the mechanical energy flux density \mathbf{F}' . These contribute a net rate of loss of energy per unit volume of $\partial F'_x / \partial x$ and the two other pairs of faces contribute $\partial F'_y / \partial y$ and $\partial F'_z / \partial z$, where F'_y and F'_z are the y - and z -components of \mathbf{F}' . The balance of energy for the element cannot be described completely in terms of fluxes across the sides. There is an additional loss of energy per unit volume of $\rho\epsilon$, where ϵ is a positive quantity called the dissipation rate.

Returning to (4.6.5), the remaining contribution to the rate of change of kinetic energy of the element is

$$-\rho\epsilon \delta x \delta y \delta z,$$

which is interpreted as the rate of loss of energy *within* the element due to viscous processes. Thus ϵ is called the rate of *dissipation* of mechanical energy per unit mass, or simply the dissipation rate.

The same ideas can be applied to a large volume of fluid that can be subdivided mentally into small volume elements like that shown in Fig. 4.6. The transfers of energy across the faces of the elements merely represent a flow of energy from one part of the fluid to another, and therefore make no contribution to the energy balance of the large volume except for the contributions from the outer surface of that volume. On the other hand, the dissipation in each volume element contributes to the total energy loss in the large volume. In other words, integration of (4.6.5) over a volume gives an equation for the rate of change of

$$K = \iiint \frac{1}{2} \rho \mathbf{u}^2 dx dy dz, \quad (4.6.6)$$

the kinetic energy of the volume of fluid, the equation being

$$dK/dt + \iint F'_n dS = - \iiint \rho\epsilon dx dy dz, \quad (4.6.7)$$

where F'_n denotes the outward normal component of the flux across the surface of the volume and dS an element of area, so that the integral is the total rate of transfer of energy across the surface. The integral on the right-hand side is over the volume concerned. For instance, the volume concerned could be that part of an ocean below some fixed level surface. Then (4.6.7) says that changes in kinetic energy result from transfer across this surface, transfer across the bottom, and dissipation within.

It is also possible to derive an equation for a large *material* volume of fluid, i.e., a volume with mobile boundaries but always consisting of the same fluid particles. An example is the ocean, which is bounded above not by a fixed surface but by its (constantly moving) free surface. As before, this volume could be subdivided into material volume elements and the balances for each element added. This is equivalent to integrating (4.6.1) over the material volume, which for a homogeneous fluid again leads to (4.6.7), with the integrations now being over the material volume and its surface. The advective contribution to the transfer across the surface is zero since by definition no advection takes place across the surface of a material volume.

In the case of a homogeneous ocean or lake, energy is transferred across the free surface from the atmosphere through the working of the pressure force and through "diffusion" of energy, which represents the action of viscous stresses. Since the normal velocity is zero at the bottom, the pressure force cannot do work there, so the only means of losing energy involve viscosity, namely, viscous stress acting on the bottom and viscous dissipation within the ocean or lake. Since the kinetic energy of such bodies of water does not continually increase, the energy losses through viscous effects must balance the energy inputs over a long period of time. At first

sight, this seems to contradict the statement that viscous effects are *not* important on the larger scales such as those characteristic of the energy inputs. The implication is that energy is transferred from one scale to another [which is possible because of the nonlinear terms in (4.5.14)] and significant dissipation takes place only at scales where velocity gradients are large enough for (4.6.2) to give values such that dissipation balances inputs. These scales are very small in practice, and can be estimated if it is assumed that the scale depends only on ϵ and ν . The only combination of these parameters with the dimensions of length is

$$(\nu^3/\epsilon)^{1/4},$$

and typical values for the ocean and atmosphere are of the order of a millimeter. [For a discussion of dissipation processes in the ocean, see Woods (1982).] This fact creates a problem for numerical models that cannot hope to cope with scales ranging from the size of the earth to the dissipation scale (ten factors of ten!). A common technique is to make the viscosity artificially large (in which case it is called an *eddy viscosity*) so that sufficient energy dissipation can occur on scales resolvable by the numerical scheme. Since the vertical resolution is usually much better than the horizontal resolution, smaller values can be used for vertical "diffusion" of momentum and energy than are used for horizontal diffusion. [Vertical eddy viscosities may be 10^7 or 10^8 times the molecular value, i.e., typical of oils rather than air or water. Horizontal eddy viscosities used are often 10^{10} or even 10^{11} times molecular, i.e., akin to very viscous substances like glucose (Weast, 1971–1972).] There is, however, no guarantee that this procedure will remove energy in a realistic way, and a major problem of numerical modeling is to find schemes that will remove energy realistically.

Energy principles are such that the loss of mechanical energy by dissipation represents a rate of *conversion* of energy into a different form, namely, heat. Thus there is a contribution $\rho\epsilon$ to the term in (4.4.4) that represents a rate of gain of internal energy per unit volume. This contribution is, however, so tiny that it is nearly always neglected. The terms $-wgp'$ and $p' \nabla \cdot \mathbf{u}$ that appear in (4.6.3) also represent rates of conversions of energy from one form to another. The former term is the product of the upward buoyancy force per unit volume $-g\rho'$ and the rate w of movement in the direction of that force, and therefore represents the rate of working per unit volume by the buoyancy force. The latter term is the product of a pressure p' and the fractional rate of change of volume [see (4.2.2)] of a material element, and therefore represents the rate of release of energy per unit volume of the fluid by expansion. The energy conversions associated with these terms are dealt with in the next section.

4.7 Total Energy Equation

Thus far, equations have been derived for energy in two forms: internal energy [Eq. (4.4.4)] and kinetic energy [Eq. (4.6.3)]. If the version of the latter equation with primes is used and the two equations are added, the term $p' \nabla \cdot \mathbf{u}$ disappears because it represents a rate of conversion of energy from the internal form to the kinetic form. The same is true of the term $\rho\epsilon$ (which cancels a contribution to Q_H). Another term

conversion
terms
 $p' \nabla \cdot \mathbf{u}$
 $\rho\epsilon$
 $-wgp'$

that requires interpretation as a rate of conversion of energy from one form to another is the buoyancy term $-wgp'$ in (4.6.3). This represents work done by gravitational forces when fluid crosses geopotential surfaces, and the appropriate form of energy per unit mass is the geopotential Φ defined by (4.5.6). With this interpretation, Φ is called the *potential* energy per unit mass associated with gravitational and centrifugal forces. Φ depends only on z , and so by definition (4.1.7) its rate of change, following a material element, is

$$D\Phi/Dt \equiv \mathbf{u} \cdot \nabla \Phi \equiv \mathbf{u} \cdot \mathbf{g} = wg \quad (4.7.1)$$

by (4.5.10). This can be converted into a rate of change for a fixed volume element by using the standard formula (4.3.6), giving

$$\partial(\rho\Phi)/\partial t + \nabla \cdot (\rho\Phi\mathbf{u}) = wgp. \quad (4.7.2)$$

The total energy equation is now obtained by adding (4.7.2), the internal energy equation (4.4.4), and the version of the kinetic energy equation (4.6.3) without primes. The result is

$$\partial(\rho(E + \Phi + \frac{1}{2}\mathbf{u}^2))/\partial t + \nabla \cdot \mathbf{F}^{\text{tot}} = Q_H, \quad (4.7.3)$$

where \mathbf{F}^{tot} is the total energy flux vector given by

$$\mathbf{F}^{\text{tot}} = \rho\mathbf{u}(E + \Phi + \frac{1}{2}\mathbf{u}^2) + p\mathbf{u} + \mathbf{F}^{\text{rad}} - k\nabla T - \mu\nabla(\frac{1}{2}\mathbf{u}^2). \quad (4.7.4)$$

The terms contributing to \mathbf{F}^{tot} in order of appearance are the advective flux, the rate of working by pressure forces per unit area, the radiative flux, the flux by diffusion of heat, and the flux by diffusion of kinetic energy. Equation (4.7.3) gives the changes for a fixed volume element. Using (4.3.6), the equation for changes of a material volume element is

$$\rho D(E + \Phi + \frac{1}{2}\mathbf{u}^2)/Dt + \nabla \cdot (p\mathbf{u} + \mathbf{F}^{\text{rad}} - k\nabla T - \mu\nabla(\frac{1}{2}\mathbf{u}^2)) = Q_H. \quad (4.7.5)$$

As with the kinetic energy equation, (4.7.3) can be interpreted in terms of balances for a fixed volume element like those illustrated in Fig. 4.6. Also, by adding contributions of many such elements, an equation for the rate of change of total energy of a large volume can be obtained. The internal energy I of the volume is defined by

$$I = \iiint \rho E \, dx \, dy \, dz, \quad (4.7.6)$$

and the potential energy P by

$$P = \iiint \rho\Phi \, dx \, dy \, dz \approx \iiint \rho g z \, dx \, dy \, dz \quad (4.7.7)$$

by (3.5.2). The total energy [see (4.6.6)] is thus

$$K + I + P,$$

whose rate of change is given by

$$d(K + I + P)/dt + \iint F_n^{\text{tot}} \, dS = \iiint Q_H \, dx \, dy \, dz. \quad (4.7.8)$$

F_n^{tot} denotes the outward component of the flux across the surface of the volume and dS an element of area, so that the integral is the total rate of transfer of energy across the surface. As found in the previous section, the equation applies both to a fixed volume, across which fluid may flow, or to a material volume, which may have a moving surface but always consists of the same particles.

The above discussion of energy is satisfactory and consistent when no phase changes (or chemical changes, etc.) occur, for then Q_H is zero. If Q_H is nonzero, it must represent a conversion of energy from one form to another. For instance, the effect of latent heat release by means of the pseudoadiabatic process can be included by using expression (4.4.10) for Q_H . Substituting in (4.7.8) and using the standard relation (4.3.6) then give

$$d(K + I + P + L)/dt + \iint (F_n^{\text{tot}} + \rho q L_v u_n) \, dS = 0, \quad (4.7.9)$$

where

$$L = \iiint L_v \rho q \, dx \, dy \, dz \quad (4.7.10)$$

is the energy that could be released by moving each parcel adiabatically upward to the saturation level and then pseudoadiabatically upward until all moisture is removed.

Estimates of the terms K , I , P , and L for the atmosphere have been made by Oort (1971) for each month of the year, and fluxes of energy across circles of latitude have also been made. The largest contributions to the mean total energy are I (73%), defined by (4.7.6) and (3.2.7), and P (25%), defined by (4.7.7). However, with these definitions, $I + P$ represents the energy that could be obtained by lowering the temperature of the atmosphere to absolute zero and bringing the mass of the atmosphere down to sea level. Since little of this energy could be obtained by a process that could readily occur, Lorenz (1955) has introduced the concept of *available potential energy* as the energy that could be obtained by some well-defined process. Usually the process considered is an adiabatic redistribution of mass without phase changes to a statically stable state of rest (see Sections 3.5 and 3.6). With this definition, the available potential energy of the atmosphere has been estimated (Price, 1975) to be about 23×10^{20} J, giving a mean over the whole earth of about 4.5×10^6 J m⁻². This may be compared with the mean available potential energy in a typical mid-latitude ocean gyre, estimated by Gill *et al.* (1974) to be of the order of 10^5 J m⁻². An alternative definition for the atmosphere [a discussion of the concept of available potential energy is given by Dutton and Johnson (1967)] would also include the energy L that could be released by condensing all the moisture. This amounts to 64×10^6 J m⁻² (Peixoto *et al.*, 1981).

For numerical models of the atmosphere and ocean, finite-difference approximations of the balances of mass, momentum, internal energy, etc., for finite volume elements are used, as discussed in Section 4.3. It does not follow automatically that a finite-difference equivalent of the total energy equation will exist. All that can be said is that the total energy equation will be satisfied, correct to a certain order, as the size of the element shrinks to zero. However, it is always possible to write the

finite-difference equations in such a way that a finite-difference form of the total energy equation is *exactly* satisfied. In other words, there are no energy sources or sinks within the body of fluid, only conversions of energy from one form to another. Such formulations have been found to eliminate problems that may be encountered otherwise, namely, an artificial increase or decrease of energy over a long period of time (Arakawa, 1966).

4.8 Bernoulli's Equation

A variant of (4.7.5) can be deduced from the identity

$$\nabla \cdot (p\mathbf{u}) = \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u} = \frac{Dp}{Dt} - \frac{\partial p}{\partial t} - \frac{p}{\rho} \frac{D\rho}{Dt},$$

which follows from the definition (4.1.7) and from the continuity equation (4.2.3). The result may also be written

$$\nabla \cdot (p\mathbf{u}) = \rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) - \frac{\partial p}{\partial t}, \quad (4.8.1)$$

and so (4.7.5) becomes

$$\rho D(E + p/\rho + \Phi + \frac{1}{2}\mathbf{u}^2)/Dt + \nabla \cdot (\mathbf{F}^{\text{rad}} - k \nabla T - \mu \nabla(\frac{1}{2}\mathbf{u}^2)) = Q_H + \partial p/\partial t. \quad (4.8.2)$$

For applications not connected with acoustic waves, the term $\partial p/\partial t$ is often relatively small, changes in pressure due to a change in level of a fluid element being large compared with changes at a fixed point. Also, viscous and diffusive effects can be ignored except on the smallest scales. Thus in situations for which radiative heating and latent heat release can also be ignored, Eq. (4.8.2) becomes

$$D(E + p/\rho + \Phi + \frac{1}{2}\mathbf{u}^2)/Dt = 0. \quad (4.8.3)$$

This is known as Bernoulli's equation since both Daniel and John Bernoulli contributed to special forms of it. [A historical discussion is given by Truesdell (1954b).] A discussion of circumstances in which it is valid may be found in Batchelor (1967; Section 3.5).

The quantity $E + p/\rho$, which appears in (4.8.3), often occurs in thermodynamics and is called the *enthalpy* per unit mass. For a perfect gas, it follows from (3.2.10) and (3.2.12) that

$$E + p/\rho = c_p T, \quad (4.8.4)$$

and this approximation is used for applications to the atmosphere. Corrections to (4.8.4) for moist air can be found in Table 85 of the Smithsonian Meteorological Tables (List, 1951). In applications to the atmosphere, the quantity $E + p/\rho + \Phi$ is sometimes called the *dry static energy* per unit mass. An approximate expression for this quantity for air is

$$E + p/\rho + \Phi \simeq c_p T + gz. \quad (4.8.5)$$

The Bernoulli equation (4.8.3) can be modified to include effects of latent heat release in a pseudoadiabatic process by using the approximation for Q_H in (4.8.2). The modified version is

$$D(E + p/\rho + \Phi + L_v q + \frac{1}{2}\mathbf{u}^2)/Dt = 0. \quad (4.8.6)$$

The quantity

$$E + p/\rho + \Phi + L_v q \simeq c_p T + gz + L_v q, \quad (4.8.7)$$

which appears in the equation, is sometimes called the *moist static energy* per unit mass.

4.9 Systematic Effects of Diffusion

The rates of diffusion of salt in water ($\kappa_D = 1.5 \times 10^{-9} \text{ m}^2 \text{ s}^{-1}$) and of water vapor in air ($\kappa_D = 2.4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$) are so small that diffusion plays no direct role on larger-scale motions. However, diffusion is systematic in that it always acts to *reduce* gradients. An equation that shows this effect can be derived from (4.3.8) in a similar way to that by which the mechanical energy equation (4.6.5) was derived from the momentum equations. Multiplication of (4.3.8) by s gives

$$\rho D(\frac{1}{2}s^2)/Dt = \nabla \cdot (\rho \kappa_D \nabla(\frac{1}{2}s^2)) - \rho \kappa_D (\nabla s)^2, \quad (4.9.1)$$

or, using the identity (4.3.6), this can be written

$$\partial(\frac{1}{2}\rho s^2)/\partial t + \nabla \cdot (\frac{1}{2}\rho s^2 \mathbf{u} - \rho \kappa_D \nabla(\frac{1}{2}s^2)) = -\rho \kappa_D (\nabla s)^2. \quad (4.9.2)$$

An alternative version has s in (4.9.1) and (4.9.2) replaced by the salinity perturbation s' , defined by

$$s' = s - s_0. \quad (4.9.3)$$

s_0 could be any constant since a constant value satisfies (4.3.8), (4.9.1), and (4.9.2) identically, but a natural choice would be the mean salinity for the volume of fluid being considered (e.g., the value for the ocean is about 0.0348 or 34.8‰; see Fig. 3.2).

Equation (4.9.2) can be given a physical interpretation similar to that illustrated in Fig. 4.6 for the energy equation. The squared salinity in a small volume is changed not only by fluxes across the surface of the volume, but also there is a systematic *loss* represented by the negative definite term on the right-hand side of (4.9.2).

If (4.9.2), or the equivalent version with primes, is integrated over the whole of the ocean, the same apparent contradiction is obtained as for that with the mechanical energy equation. There is an input across the ocean surface at large scale [because surface salinity tends to be high where there is a salt flux into the ocean—see, e.g., Defant (1961, Volume 1, Fig. 68)], but losses by diffusion are insignificant at the large scales. As with energy, transfers from one scale to another take place because of the nonlinear advection terms in (4.3.8), the significant contributions to the right-hand side of (4.9.2) coming at very small scales. On this basis, Stern (1968) estimated the root-mean-square salinity gradient in the upper ocean to be about 1000 times the mean value.