The heat and salinity equations (4.10.3) and (4.10.4) can be expressed in terms of λ, φ, r coordinates by using the expressions (4.12.9) and (4.12.10) and the fact that the components of the gradient vector $\nabla\gamma$ are

$$\nabla\gamma \equiv \left(\frac{1}{r\cos\varphi}\frac{\partial\gamma}{\partial\lambda}, \frac{1}{r}\frac{\partial\gamma}{\partial\varphi}, \frac{\partial\gamma}{\partial z}\right). \tag{4.12.22}$$

Chapter Five

Adjustment under Gravity in a Nonrotating System

5.1 Introduction: Adjustment to Equilibrium

The first two chapters sought to give some insight into the way energy from the sun is absorbed by the atmosphere–ocean system, how fluid motion results, and how this motion affects the mean distribution of temperature. As Halley (1686, p. 165) stated (see Chapter 2), the system is driven by "the Action of the Suns Beams upon the Air and Water," and so "according to the Laws of *Staticks*, the Air which is less rarified or expanded by heat, and consequently more ponderous, must have a Motion towards those parts thereof, which are more rarified, and less ponderous, to bring it to an *Æquilibrium*." This chapter marks the beginning of a more detailed study of the way the atmosphere–ocean system *tends to adjust* to equilibrium. The adjustment processes are most easily understood in the absence of driving forces. Suppose, for instance, that the sun is "switched off," leaving the atmosphere and ocean with some nonequilibrium distribution of properties. How will they respond to the gravitational restoring force? Presumably there will be an adjustment to some sort of equilibrium. If so, what is the nature of the equilibrium? How long does the adjustment take? In what way is the adjustment process most readily described and understood?

The problem will be studied in stages, roughly following the historical development. In this chapter, for instance, complications due to the rotation and shape of the earth will be ignored and only small departures from the hydrostatic equilibrium of Section 3.5 will be considered. The nature of the adjustment processes will be found by deduction from the equations of motion developed in Chapters 3 and 4.

This method was not available in the seventeenth century, but it was possible instead to study simpler systems in the laboratory and thereby gain an improved understanding of Nature. A remarkable example is found in the work of Marsigli (1681). It seems (Deacon, 1971, pp. 147–149) that when Marsigli went to Constantinople in 1679 he was told about an undercurrent in the Bosphorus that was well-known to local fishermen. The undercurrent was in fact referred to in a sixth century discussion of flows through straits by Procopius of Caesarea (History of the Wars VIII, vi. 27) "... for the fishermen of the towns on the Bosphorus say that the whole stream does not flow in the direction of Byzantium, but while the upper current which we can see plainly does flow in this direction, the deep water of the abyss, as it is called, moves in a direction exactly opposite to that of the upper current and so flows continuously against the current which is seen." [That is, the undercurrent flows toward the Black Sea from the Mediterranean. Defant (1961, Chapter 16) gives a modern description.] By observing the distortions and feel of a rope lowered into the water, Marsigli found that the current reversal occurred at depths varying between 8 and 12 Turkish feet. He reasoned that the effect was due to density differences, and so made measurements of these differences using a hydrostatic balance. He found that water from the Black Sea is lighter than water from the Mediterranean, giving readings on his instrument up to $29\frac{1}{4}$ grains lower. He attributed the low density of the Black Sea to lower salinity resulting from river runoff. Marsigli then measured the density of samples taken from the surface of the Bosphorus and from the undercurrent. Here the difference was 10 grains, values being consistent with a Mediterranean origin for the undercurrent and a Black Sea origin for the surface water. To clinch the point, he performed a laboratory experiment, which he illustrated as shown in Fig. 5.1. A tank

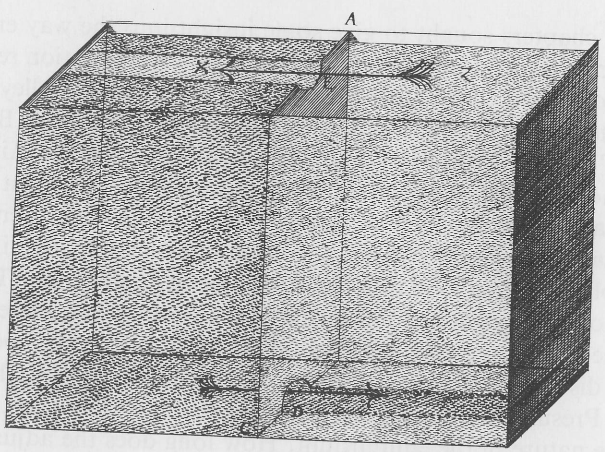

Fig. 5.1. A figure from Marsigli (1681) illustrating adjustment under gravity of two fluids of different density. Initially the container was divided in two by a partition. Side *X* contained water taken from the undercurrent in the Bosphorus. Side *Z* contained dyed water having the density of surface water in the Black Sea. The experiment was to put holes in the partition at *D* and *E* and to observe the resulting flow. The flow through the lower hole was in the direction of the undercurrent in the Bosphorus, while the flow through the upper hole was in the direction of the surface flow.

was divided in two by a partition. Side *X* was filled with water taken from the under-current and side *Z* with dyed water having density equal to that of the Black Sea. Holes in the partition at *D* and *E* were then opened and water from *X* was observed to pass through hole *D* to side *Z*, whereas the movement through hole *E* was in the opposite direction. Marsigli noted that this internal adjustment could give surface currents in the observed direction without requiring a difference in sea level between the Black Sea and the Mediterranean. He had already attempted to measure such a difference using a mercury barometer.

The force that produces the motion when the holes in Marsigli's apparatus are opened is gravity, which produces pressure differences between the two sides. An arrangement that will receive further study in this chapter is the one shown in Fig. 5.2a. A fluid of uniform density ρ_c is at rest on two sides of a partition, the surface levels on the two sides differeing by an amount h. By (3.5.8), the pressure at A exceeds the pressure at B by an amount $\rho_c g h$, so that if the partition is suddenly removed, the fluid near the partition will start toward the right. A situation closer to that of Marsigli's experiment is illustrated in Fig. 5.2b, where now there are two fluids, one of smaller density ρ_1 and the other of larger density ρ_2. A partition divides the lower fluid into two parts as shown. The upper fluid is in equilibrium and is assumed *not* to be completely divided by the partition. It follows that the pressures at any level above the top of the partition are equal, and the same is true at the level of C and D by the hydrostatic equation (3.5.8). However, the same equation shows that the pressure difference between C and A exceeds the difference between D and B by an amount $(\rho_2 - \rho_1)gh$, so the pressure difference between A and B is $(\rho_2 - \rho_1)gh$. When the partition is removed, motion toward the right will take place as in case (a), but the adjustment is slower since the pressure difference is reduced by a factor $(\rho_2 - \rho_1)/\rho_2$

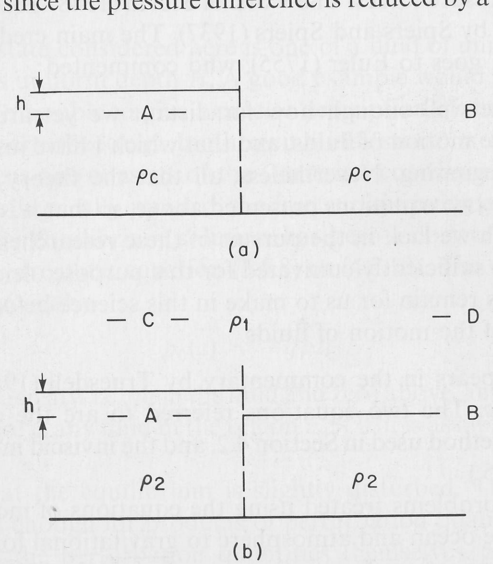

Fig. 5.2. Initial states considered for problems of adjustment under gravity. The dashed line marks a partiti (of finite height) which is removed at the initial instant. (a) The fluid of density ρ_c has different depths on the t sides and is bounded above by a free surface. The difference h in depths is supposed to be very small, so th the problem to be solved is linear. (b) The only difference is that a fluid of density $\rho_1 < \rho_2$ lies above the lower flu

The adjustment processes are, in fact, exactly the same as they would be in case (a) if the gravitational acceleration were reduced to a value g' given by

$$g' = g(\rho_2 - \rho_1)/\rho_2. \tag{5.1.1}$$

g' is called *reduced gravity* for this reason. This example also shows that the driving force is proportional to

$$(\rho_2 - \rho_1)g,$$

the density difference times g. This product is called the *buoyancy* force per unit volume (see Section 4.5).

In the quantitative treatment of this problem, which will be developed in this chapter, the difference h in initial levels will be assumed to be small. This simplifies the mathematics because it leads to a linear problem, and so solutions can be superposed. This chapter is concerned with adjustment of a homogeneous fluid with a free surface (including the case depicted in Fig. 5.2a). Chapter 6 deals with internal adjustment of a density-stratified fluid, including problems such as that illustrated in Fig. 5.2b.

Although the qualitative ideas in the above argument (relating to Fig. 5.2) can be found in the work of Archimedes (287–212 BC) ["On Floating Bodies," English translation in Hutchins (1952); see also discussion in Dugas (1957)], quantitative treatment of the problem required first the development of the laws of motion and of the calculus needed to apply the laws. Both of these developments are among the achievements of Newton (whose *Principia* was printed in 1687 with the help and encouragement of Halley). Also required was a proper understanding of hydrostatics and the nature of pressure forces. The work of Stevin (1548–1620) and Pascal on this subject is translated by Spiers and Spiers (1937). The main credit for developing the equations of motion goes to Euler (1755), who commented:

> But here we see well enough how far distant we yet are from the complete knowledge of the motion of fluids, and that which I have just explained contains only a feeble beginning. Nevertheless, all that the theory of fluids includes is contained in the two equations presented above, so that it is not the principles of mechanics which we lack in the pursuit of these researches, but solely analysis, which is not yet sufficiently cultivated for this purpose. And thus we see clearly what discoveries remain for us to make in this science before we can arrive at a perfect theory of the motion of fluids.

(This translation appears in the commentary by Truesdell (1954b, p. LXXXIX) in Euler's *Opera Omnia.* The two equations referred to are the continuity equation, derived by the first method used in Section 4.2, and the inviscid momentum equations, derived in Section 4.5.)

Among the first problems treated using the equations of motion were problems of the response of the ocean and atmosphere to gravitational forces. Laplace (1778–1779) developed the equations for motion on a rotating sphere under the action of tide-generating forces and found solutions for the "equilibrium" tide in a constant-depth worldwide ocean. He also encountered the problem of treating thermal forcing of the atmosphere.

> Our atmosphere consists of an elastic fluid whose density is a function of pressure and temperature. These are not constant at a given point in the atmosphere because the rotating earth presents a different point to the sun at each instant of the day, and, because of the inclination of the ecliptic, each day has a different length and the elevation of the sun increases or decreases. It is readily seen that the variations in heating due to these different causes must excite oscillations which it seems impossible to submit to calculation because the law of these variations . . . has not been sufficiently well determined.

[My paraphrasing; see "On oscillations of the atmosphere," Laplace "Œuvres" (1893, pp. 283–301).] However, by making some assumptions Laplace reduced the problem to one of forced oscillations of an isothermal atmosphere, which he found to obey the tidal equations already deduced for the ocean, but with the ocean depth replaced by the scale height of the atmosphere, which he calculated to be 27,000 ft (the surface pressure in feet of water multiplied by the density ratio between air and water, i.e. 32 × 850).

The final part of Laplace's paper ["On waves" (Laplace, "Œuvres" (1893, pp. 301–310))] treats the problem to be dealt with in the next two sections, namely, the adjustment of a homogeneous fluid of constant depth that initially has a small displacement of its free surface. He found, in particular, the relation (5.3.8), which showed that disturbances propagate away from the disturbed region at a speed that depends on the curvature of the surface.

5.2 Perturbations from the Rest State for a Homogeneous Inviscid Fluid

The equilibrium state considered here is one of a fluid of uniform density ρ_c that is at rest and that has uniform depth H. A good example would be an artificial pond with a flat bottom. To be able to give a precise description of the motion that occurs when the system is perturbed (e.g., by throwing in a stone), a coordinate system is required. A convenient one consists of the Cartesian coordinates (x, y, z) chosen so that the z axis points vertically upward, the free surface being at $z = 0$ and the bottom at $z = -H$. The equilibrium solution has zero velocity, and the pressure is determined by the hydrostatic equation (3.5.8) or (4.5.18). The equilibrium pressure $p_0(z)$ in this case is given by

$$p_0(z) = -g\rho z, \tag{5.2.1}$$

where ρ is the in-situ density, i.e., ρ_c in the fluid and zero above, and g is the acceleration due to gravity. (If there is any fluid in the region $z > 0$, it is assumed to have negligible density.)

Suppose now that the equilibrium is slightly disturbed. The perturbations are assumed to be small enough for products of perturbation quantities to be neglected in comparison with the perturbation quantities themselves. Suppose that (u, v, w) are the velocity components corresponding to the coordinates (x, y, z) and that the disturbed position of the free surface (see Fig. 5.3) is given by

$$z = \eta(x, y, t). \tag{5.2.2}$$

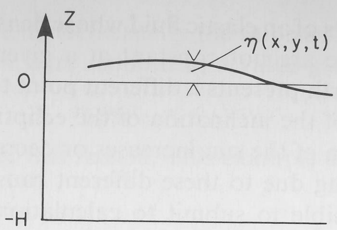

Fig. 5.3. The geometry of the disturbed surface. The displacement from the rest position is η and the undisturbed depth is H.

For this problem, it is convenient to define the perturbation pressure by

$$p = -g\rho z + p', \tag{5.2.3}$$

where ρ is the insitu density, i.e., ρ_c in the fluid and zero above. (This differs from the definition of Section 4.5 only in the infinitesimal region between the disturbed and undisturbed positions of the free surface.)

The equations of motion consist of the continuity equation (4.2.3) and the momentum equations (4.5.7)–(4.5.9) for an inviscid fluid. Since the density is constant within the fluid, the rotation rate $\mathbf{\Omega}$ is 0, and products of perturbation quantities can be neglected, the continuity equation is in this case

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \tag{5.2.4}$$

and the momentum equation is

$$\rho\, \partial u/\partial t = -\partial p'/\partial x, \qquad \rho\, \partial v/\partial t = -\partial p'/\partial y, \tag{5.2.5}$$

$$\rho\, \partial w/\partial t = -\partial p'/\partial z. \tag{5.2.6}$$

Adding the x, y, and z derivatives of the above three components of the momentum equation, and using the continuity equation (5.2.4), there results an equation for p', namely, Laplace's equation

$$\nabla^2 p' \equiv \partial^2 p'/\partial x^2 + \partial^2 p'/\partial y^2 + \partial^2 p'/\partial z^2 = 0. \tag{5.2.7}$$

(In connection with the present problem, Laplace "Œuvres" (1893, pp. 301–310) found this as an equation for the vertical displacement of a material particle). The condition (see Section 4.11) that must be satisfied at the bottom, where $Z = -H$, is one of no normal flow, i.e.,

$$w = 0 \quad \text{at} \quad z = -H. \tag{5.2.8}$$

The condition that a particle in the free surface $z = \eta$ will remain in it (see Section 4.11.1) is in this case

$$D(z - \eta)/Dt = 0,$$

i.e.,

$$w = \partial\eta/\partial t + u\, \partial\eta/\partial x + v\, \partial\eta/\partial y, \tag{5.2.9}$$

which, for small perturbations, reduces to

$$w = \partial\eta/\partial t \quad \text{at} \quad z = \eta. \tag{5.2.10}$$

In addition, the pressure must vanish at the free surface, i.e.,

$$p = p_0 + p' = 0 \quad \text{or} \quad p' = \rho g \eta \quad \text{at} \quad z = \eta \tag{5.2.11}$$

by (5.2.3). Also, since the differences in the solutions for w and p' between $z = \eta$ and $z = 0$ are small, Eqs. (5.2.10) and (5.2.11) can both be applied at $z = 0$ and will be correct to the same order of approximation.

The problem is to solve Laplace's equation (5.2.7) subject to the boundary conditions (5.2.8) at the bottom and (5.2.10) and (5.2.11) at $z = 0$. There are in fact a great variety of solutions depending on the initial condition, i.e., the nature of the perturbation at the beginning. In the next section, solutions will be considered where p' varies *sinusoidally* with horizontal position. This is not a real restriction because an arbitrary disturbance can be described as a superposition of such waves by Fourier's theorem.

5.3 Surface Gravity Waves

A disturbance that is sinusoidal in the horizontal can take the form of a traveling wave or of a standing wave. In particular, a "long-crested" traveling wave has the form

$$\eta = \eta_0 \cos(kx + ly - \omega t), \tag{5.3.1}$$

where η_0 is the *amplitude*, the vector

$$\mathbf{k} = (k, l)$$

is the *wavenumber* (proportional to the number of waves per unit distance), ω is the *frequency*, and the quantity

$$\hat{\Phi} = kx + ly - \omega t = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{5.3.2}$$

is called the *phase* of the wave. Such a wave consists of a sinusoidal corrugation of the surface that moves at uniform speed. A sketch of the wave is shown in Fig. 5.4 with sections cut normal to the wave crests and along the x axis. In the section normal to the wave crests, one sees a series of waves with wavelength $2\pi\kappa^{-1}$ where κ, given by

$$\kappa^2 = k^2 + l^2, \tag{5.3.3}$$

is the magnitude of the wavenumber. In this plane, the crests move at a speed

$$c = \omega/\kappa, \tag{5.3.4}$$

called the *phase speed* (i.e., the speed of lines of constant phase $\hat{\Phi}$). Their rate of movement in any other plane *appears* to be faster by a factor equal to the secant of the angle between that plane and the plane normal to the crests. For instance, in Fig. 5.4 the cut along the x axis shows a greater apparent wavelength $2\pi k^{-1}$ than that for the plane normal to the crests, and the apparent propagation speed is proportionately higher. This should be borne in mind when propagation is observed along only one plane.

Although one is free to choose the initial disturbance to be sinusoidal in the horizontal, it does not follow that the traveling wave (5.3.1) is a possible form of

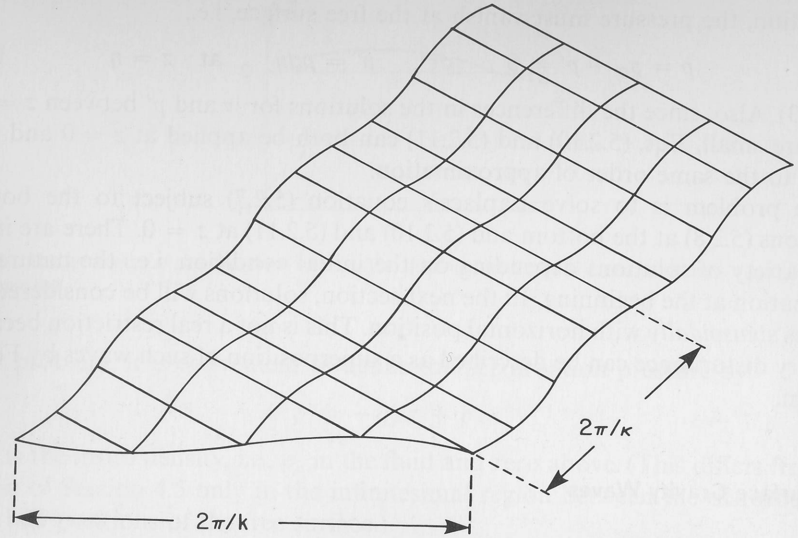

Fig. 5.4. A plane sinusoidal wave train moving at an angle to the axis. The wavenumber (k, l) has magnitude κ. Note that the wavelength $2\pi/k$ observed in a section along the x axis is larger than the actual wavelength $2\pi/\kappa$.

motion. This can be deduced only from the equations. If, however, it is assumed that p' is proportional to η, as given by (5.3.1), Laplace's equation (5.2.7) gives

$$\partial^2 p'/\partial z^2 - \kappa^2 p' = 0, \qquad (5.3.5)$$

so that, at a given horizontal position and time, the vertical variation of p' must be a sum of exponentials or hyperbolic functions. The boundary condition (5.2.8), together with (5.2.6), shows that $\partial p'/\partial z$ must vanish at $z = -H$. Since p' is also given by (5.2.11) at $z = 0$, the solution must be

$$p' = \frac{\rho g \eta_0 \cos(kx + ly - \omega t) \cosh \kappa(z + H)}{\cosh \kappa H}, \qquad (5.3.6)$$

with the vertical velocity component [see (5.2.6)] given by

$$w = \frac{\kappa g \eta_0 \sin(kx + ly - \omega t) \sinh \kappa(z + H)}{\omega \cosh \kappa H}. \qquad (5.3.7)$$

It remains to satisfy condition (5.2.10) at $z = 0$. Substitution shows that this *is* consistent with the assumed form (5.3.1), provided that

$$\omega^2 = g\kappa \tanh \kappa H. \qquad (5.3.8)$$

This important equation determines the frequency and hence the phase speed of waves of a given wavenumber, such an equation being called a *dispersion relation*. The above dispersion relation was obtained by Laplace "Œuvres" (1893, pp. 301–310). Figure 5.5 shows graphs of ω and $c = \omega/\kappa$ as functions of κ.

One important property is that the frequency does not depend on the *direction* of the wave, but only on the *magnitude* of the wavenumber. Thus waves of a given

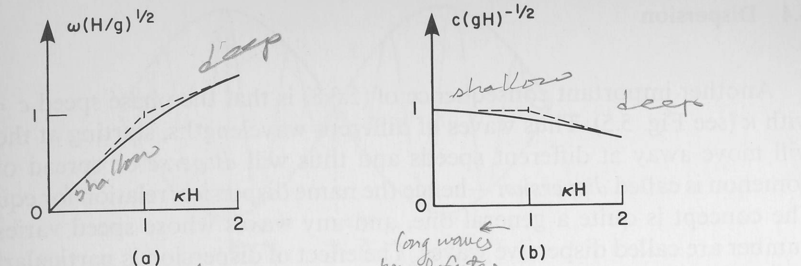

Fig. 5.5. The dispersion relation for surface gravity waves on water of depth H. (a) Frequency ω and (b) phase speed c as functions of wavenumber κ. The dashed line shows the long-wave approximation for $\kappa H < 1$ and the short-wave approximation for $\kappa H > 1$. The maximum error in these approximations is 13% at $\kappa H = 1$.

wavenumber that move in different directions all do so at the same speed. Consider the sort of perturbation that may be obtained by the superposition of such waves. For instance, the wave given by

$$\eta = \eta_0[\cos(kx + ly - \omega t) + \cos(kx - ly - \omega t)] = 2\eta_0 \cos ly \cos(kx - \omega t) \qquad (5.3.9)$$

represents a wave with crests parallel to the y axis that moves in the x direction with speed ω/k (faster than ω/κ). The height varies along the crest with wavelength $2\pi/l$. Another example is the standing wave

$$\eta = \eta_0[\cos(kx + ly - \omega t) + \cos(kx + ly + \omega t)] = 2\eta_0 \cos(kx + ly) \cos \omega t \qquad (5.3.10)$$

for which the wave crests remain stationary, but the surface moves up and down with frequency ω. For each wave form, the velocity field can be calculated from (5.2.5) and (5.2.6). Figure 5.6 shows how velocities relate to the free surface for (a) a traveling wave and (b) a standing wave.

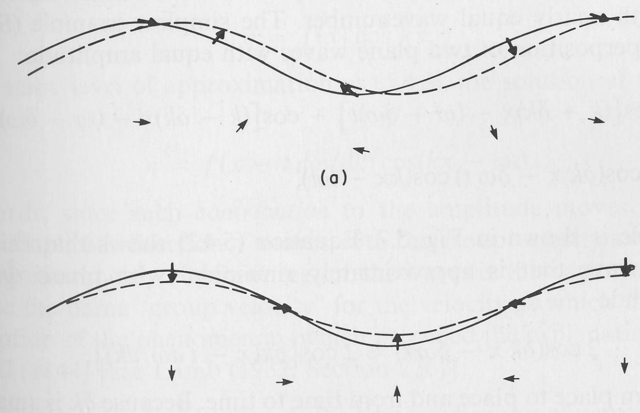

Fig. 5.6. The motion, shown by arrows, of fluid particles associated with a traveling wave (a) and a standin wave (b). The solid line shows the free surface at some initial time and the dotted line shows the position of th surface a short time later. The arrows mark particle displacements in this time. For the standing wave, the particl paths are straight-line segments whose orientation depends on position relative to the crests. For the travelin wave the particle paths are ellipses that become circular for large κH and straight line segments for small κH. I each case, the perturbation pressure is highest below crests and lowest below troughs.

5.4 Dispersion

Another important consequence of (5.3.8) is that the phase speed $c = \omega/\kappa$ varies with κ (see Fig. 5.5). Thus waves of different wavelengths, starting at the same place, will move away at different speeds and thus will *disperse* or spread out. The phenomenon is called *dispersion*—hence the name dispersion relation for equation (5.3.8). The concept is quite a general one, and any waves whose speed varies with wavenumber are called dispersive waves. The effect of dispersion is particularly noticeable with ocean waves that are generated by a distant storm (Barber and Ursell, 1948). Since long waves (small κ) travel fastest, these arrive first and may precede shorter waves from the same storm by one or two days. The fact that waves of different length become separated and arrive at different times explains why swell is so regular compared with waves produced by local winds.

The dispersion effect has been used to identify the point of origin of waves that have traveled extraordinarily large distances (Snodgrass *et al*., 1966). One set of waves observed in the North Pacific was estimated to have traveled halfway around the world from the Indian Ocean, the great circle route passing south of Australia. The direction of travel is determined from the orientation (in deep water) of the wave crests, and the distance is calculated from the difference in arrival time of waves of different length, and hence of different frequency. The dominant frequency increases progressively with time as the progressively shorter waves arrive, and the rate of change of this frequency gives the distance of travel.

Despite the effects of dispersion, in practice waves are never purely sinusoidal, but are instead a mixture of waves of different wavenumber. As a wave train travels away from its source region, the waves at a particular point become more "pure" in the sense that the wavenumbers that give significant contributions to the wave become confined to a narrower band. Hence there is particular interest in waves made up of components with nearly equal wavenumber. The simplest example (Stokes, 1876) consists of a superposition of two plane waves with equal amplitude:

$$\eta = \cos[(k + \delta k)x - (\omega + \delta\omega)t] + \cos[(k - \delta k)x - (\omega - \delta\omega)t], \quad (5.4.1)$$

i.e.,

$$\eta = 2\cos(\delta k\, x - \delta\omega\, t)\cos(kx - \omega t), \quad (5.4.2)$$

and this example is shown in Fig. 5.7. Equation (5.4.2) shows that this can be interpreted as a wave that is approximately sinusoidal with phase $\hat{\Phi} = kx - \omega t$ but with amplitude

$$2\cos(\delta k\, x - \delta\omega\, t) \approx 2\cos[\delta k(x - t\, d\omega/dk)], \quad (5.4.3)$$

which varies from place to place and from time to time. Because δk is small, however, the amplitude change from one wave crest to the next is only slight, and so (5.4.2) is an example of what is called a "slowly varying wave train." Individual wave crests move with the phase speed ω/k, but the region in which waves have large amplitudes moves with speed

$$c_g = d\omega/dk, \quad (5.4.4)$$

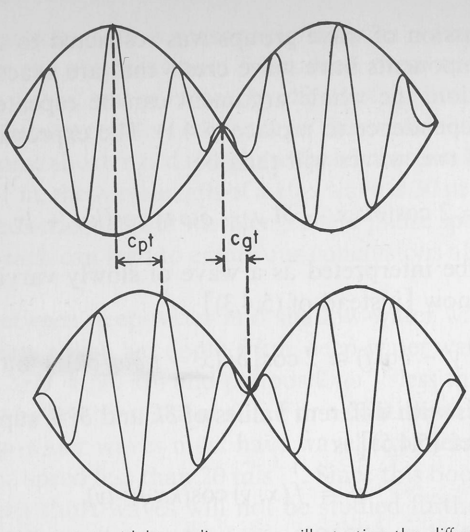

Fig. 5.7. A superposition of two sinusoidal traveling waves, illustrating the difference between the speed c_p of the wave crests and the speed c_g of the envelope of the waves, i.e., of the regions of large amplitude. The group velocity c_g equals $d\omega/dk$, which in this case is equal to $\frac{1}{2}c_p$ as for deep-water waves.

as given by (5.4.3). This speed is called the *group velocity*. It depends on the *derivative* of ω because the region of large amplitudes occurs where the phase *difference* between these two component waves has a certain value.

A more general example (Landau and Lifshitz, 1959, Section 66) consists of a superposition of many waves with wavenumber close to k, but with a range of different values of δk. This combination also may be considered as a superposition of solutions of the form (5.4.2) with a common factor $\cos(kx - \omega t)$. It follows that if the initial superposition has the form

$$\eta = f(x)\cos kx, \quad (5.4.5)$$

then, to the same level of approximation as (5.4.3), the solution at time t will have the form

$$\eta = f(x - t\, d\omega/dk)\cos(kx - \omega t). \quad (5.4.6)$$

In other words, since each *contribution* to the amplitude moves with the group velocity, the amplitude function f moves with the group velocity as well. In cases in which f is significant only in a finite region, the waves in this region are called a *wave group*. Hence the name "group velocity" for the velocity at which the group moves.

A description of the phenomenon (which preceded the explanation) was given by Scott Russell (1844) [see Lamb (1932, Section 236)]:

> It has often been noticed that, when an isolated group of waves, of sensibly the same length, is advancing over relatively deep water, the velocity of the group as a whole is less than that of the individual waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, whilst its former place in the group is occupied in succession by other waves which have come forward from the rear.

The above discussion of wave groups was restricted to a rather special case in which all of the components have wave crests that are exactly parallel. In order to remove this restriction, the whole argument can be repeated, but beginning with two waves with y-dependence to replace (5.4.1). The expression replacing (5.4.2) for the superposition of two waves will then be

$$\eta = 2\cos(\delta k\, x + \delta l\, y - \delta\omega\, t)\cos(kx + ly - \omega t). \tag{5.4.7}$$

As before, this can be interpreted as a wave of slowly varying amplitude, and the amplitude factor is now [instead of (5.4.3)]

$$2\cos(\delta k\, x + \delta l\, y - \delta\omega\, t) \approx 2\cos[\delta k(x - t\,\partial\omega/\partial k) + \delta l(y - t\,\partial\omega/\partial l)]. \tag{5.4.8}$$

If a set of such waves with different values of δk and δl is superposed and the initial form of the waves [cf. (5.4.5)] is

$$\eta = f(x, y)\cos(kx + ly), \tag{5.4.9}$$

then, to the level of approximation of (5.4.8), the form at time t will be

$$\eta = f(x - t\,\partial\omega/\partial k, y - t\,\partial\omega/\partial l)\cos(kx + ly - \omega t). \tag{5.4.10}$$

The velocity $\mathbf{c}_g$ of translation of the group is therefore the *vector quantity*

$$\mathbf{c}_g = (\partial\omega/\partial k, \partial\omega/\partial l), \tag{5.4.11}$$

i.e., $\mathbf{c}_g$ is the *gradient* of the frequency ω in the wavenumber plane. The concept of group velocity is quite general since the above argument makes no reference to any particular type of wave, and it will be useful again and again in future chapters. Further discussion is given by Lighthill (1965, 1978).

5.5 Short-Wave and Long-Wave Approximations

The length scale that appears in the dispersion relation (5.3.8) and hence determines the character of the waves is the fluid depth H. Different approximations apply, depending on how κ^{-1} relates to H. For the case of *short waves*, i.e., for $\kappa^{-1} \ll H$, (5.3.8) is approximated by (see dashed line in Fig. 5.5)

$$\omega^2 = g\kappa \tag{5.5.1}$$

and (5.3.6) by

$$p' = \rho g \eta_0 \cos(kx + ly - \omega t)\exp(\kappa z). \tag{5.5.2}$$

These are also called *deep-water* waves because $H \gg \kappa^{-1}$. The pressure perturbation and the motion are confined to a distance of order κ^{-1} from the surface, so propagation is unaffected by the bottom. For instance, the dominant waves that one sees in the ocean have periods $2\pi\omega^{-1}$ of order 10 s. By (5.5.1) a deep-water wave of period 10 s has a wavelength $2\pi\kappa^{-1}$ of about 150 m, its amplitude has an e-folding depth of 25 m, and the phase speed is 15 m s^{-1}. [Such a phase speed is typical because it matches the wind speeds found near the water surface in the generation regions, and the period follows by (5.5.1).]

The deep-water approximation is reasonable for such waves when the depth is greater than 25 m. Since the ocean is about 5-km deep, these waves move over large distances as deep-water waves, and only feel the effects of the bottom when they come near the shore. The frequency remains constant as they move into shallow water, so by (5.3.8) the waves become shorter and the phase speed decreases. For instance, when the depth is reduced to 1 m, the wavelength of a 10-s wave is 30 m and the phase speed is only 3 m s^{-1}. Thus deductions about wavelength and phase speed that are made by observing waves on a beach can lead to erroneous conclusions about their properties in deep water.

The dividing line between deep-water and shallow-water waves depends on the depth. For the deep ocean, which has depth 5 km, deep-water waves must have wavelength $2\pi\kappa^{-1}$ less than $2\pi H \approx 30$ km and periods $2\pi\omega^{-1}$ less than $2\pi(g^{-1}H)^{1/2} \approx 2$ min. Phase speeds must be less than 200 m s^{-1}. For a continental shelf of depth 50 m, on the other hand, deep-water waves must have wavelength less than 300 m, period less than 15 s, and phase speed less than 20 m s^{-1}. Since this book is primarily about large-scale motions, such short waves will not be studied further, and the reader is referred for additional information to Kinsman (1965), Lamb (1932), O. M. Phillips (1977), and Stoker (1957).

The approximation to (5.3.8) for *long waves*, i.e., for $\kappa^{-1} \gg H$, is (see dashed line in Fig. 5.5)

$$\omega^2 = g\kappa^2 H, \tag{5.5.3}$$

i.e.,

$$c^2 = gH. \tag{5.5.4}$$

These are also called *shallow-water* waves because $H \ll \kappa^{-1}$, and they are *nondispersive* because the phase speed c does not depend on wavenumber. This speed is about 200 m s^{-1} for deep water, i.e., such waves could cross the Atlantic Ocean in 7 hr. The speed on a continental shelf of depth 50 m is less by a factor of ten, i.e., about 20 m s^{-1}. The corresponding approximation to (5.3.6) is

$$p' = \rho g \eta_0 \cos(kx + ly - \omega t), \tag{5.5.5}$$

i.e., the pressure perturbation is independent of depth. Since the density perturbation is zero, this is precisely the result that would be obtained if the pressure were calculated from the hydrostatic equation (3.5.5). It will be shown in the next section that if it is assumed that the pressure is approximately equal to that given by the hydrostatic equation (called the hydrostatic approximation), (5.5.3) is obtained as well as (5.5.5). In other words, in this case at least, the *hydrostatic approximation and the long-wave (or shallow-water) approximation are equivalent*. Note also that in the small κH limit, Eq. (5.3.7) for w shows that the vertical velocity increases linearly with z from zero at the bottom to a maximum of $\partial\eta/\partial t$ at the surface.

5.6 Shallow-Water Equations Derived Using the Hydrostatic Approximation

The emphasis of this book is on motions with horizontal scale large enough compared with the vertical scale for the hydrostatic approximation to be valid. In this section, the pressure is assumed at the outset to satisfy the hydrostatic equation

$$\partial p/\partial z = -\rho g. \tag{5.6.1}$$

This leads to simplifications in the treatment of the equations, and the result is found to be the same as that obtained by applying the limit $\kappa H \to 0$ to the more general solution.

For a homogeneous fluid, (5.6.1) implies that the perturbation pressure p' satisfies

$$\partial p'/\partial z = 0, \tag{5.6.2}$$

and so the boundary condition (5.2.11) at the surface implies

$$p' = \rho g \eta \tag{5.6.3}$$

at all points within the fluid [in agreement with (5.5.5)]. The momentum equations (5.2.5) therefore become

$$\partial u/\partial t = -g\, \partial\eta/\partial x, \tag{5.6.4}$$

$$\partial v/\partial t = -g\, \partial\eta/\partial y, \tag{5.6.5}$$

showing that time-varying currents are independent of depth. This simplifies the continuity equation (5.2.4), which can now be integrated with respect to depth, using as boundary conditions (5.2.8) and (5.2.10). The result is

$$\partial\eta/\partial t + H(\partial u/\partial x + \partial v/\partial y) = 0. \tag{5.6.6}$$

The quantity $(\partial u/\partial x + \partial v/\partial y)$ is called the *horizontal divergence*, being the divergence of the horizontal component of the velocity.

The continuity equation can also be derived from first principles by considering the fluid column above a fixed element of area, as shown in Fig. 5.8. Suppose (u, v) is the velocity at the center of the element, and η the surface elevation there. Since (u, v) is independent of depth, the rate of mass flux across the central section normal to the x axis is ρu times the area $(H + \eta)\,\delta y$ of the section. The difference between the

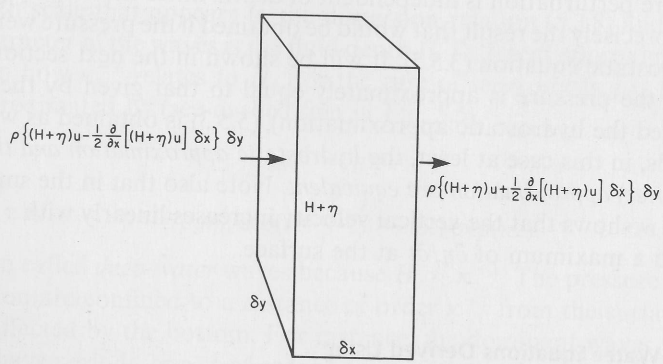

Fig. 5.8. The mass balance for a fluid column of area $\delta x\,\delta y$ when the horizontal velocity components u, v are independent of depth. The mass fluxes across two of the planes are shown.

outward flux from the right-hand face and the inward flux across the left-hand face is therefore, to the appropriate level of approximation,

$$\delta x\, \delta y\, \partial(\rho u(H + \eta))/\partial x.$$

Taking account of the other two sides and equating the net rate of inflow to the rate of change of the total mass $\rho(H + \eta)\,\delta x\,\delta y$ then give

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}[(H + \eta)u] + \frac{\partial}{\partial y}[(H + \eta)v] = 0. \tag{5.6.7}$$

This is valid even for large perturbations, provided that the horizontal velocity components u and v are independent of depth. Equation (5.6.7) can in fact be derived by integrating (5.2.4), using (5.2.8) and (5.2.9) as boundary conditions. If the perturbation is small, (5.6.7) reduces to the linear equation

$$\partial\eta/\partial t + \partial(Hu)/\partial x + \partial(Hv)/\partial y = 0, \tag{5.6.8}$$

which in turn reduces to (5.6.6) when H is constant.

An equation with only one dependent variable η can be obtained by eliminating u, v from (5.6.4), (5.6.5), and (5.6.8). The result is (Lagrange, 1781)

$$\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial x}\left(gH\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(gH\frac{\partial\eta}{\partial y}\right). \tag{5.6.9}$$

In the particular case of constant depth, this may be written in the form

$$\partial^2\eta/\partial t^2 = c^2(\partial^2\eta/\partial x^2 + \partial^2\eta/\partial y^2) \equiv c^2\,\nabla^2\eta \tag{5.6.10}$$

where c^2 is given by (5.5.4). This is the wave equation which has solutions of the form (5.3.1), showing that the hydrostatic approximation leads to the same results as the long-wave approximation. Also, as Lagrange (1781) pointed out, (5.6.10) is the same as the equation for sound propagation, so there is a complete analog between small-amplitude shallow-water waves and small-amplitude sound waves in two dimensions.

The wave equation (5.6.10) has very simple solutions when there is no dependence on y. In particular, if the fluid is initially at rest and has surface displacement

$$\eta = G(x),$$

then the solution of (5.6.10) is

$$\eta = \tfrac{1}{2}[G(x + ct) + G(x - ct)]. \tag{5.6.11}$$

The corresponding fluid velocity distribution obtained from (5.6.4) is

$$u = -\tfrac{1}{2}c^{-1}g[G(x + ct) - G(x - ct)]. \tag{5.6.12}$$

Figure 5.9 shows two special cases. (These will be contrasted in Chapter 7 with the corresponding solutions in a rotating system.) Case (a) has initial displacement the same as that in Fig. 5.2a, namely,

$$\eta = -\eta_0\,\mathrm{sgn}(x), \tag{5.6.13}$$

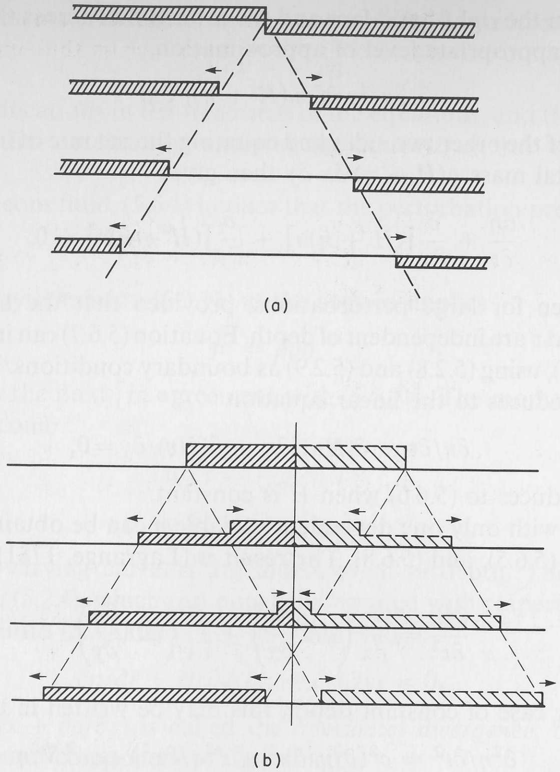

Fig. 5.9. Solutions of the shallow-water wave equation for two different initial surface displacements. In case (a) waves move out from the initial discontinuity with speed c, leaving behind zero displacement but a steady motion from right to left with velocity $c^{-1}g\eta_0$, where η_0 is the magnitude of the initial surface elevation. In case (b) there are two pairs of wave fronts. The velocity is zero everywhere except where the surface elevation is $\frac{1}{2}\eta_0$, where η_0 is the initial displacement at the center. In these places, the velocity is $\frac{1}{2}c^{-1}g\eta_0$ and directed away from the axis of symmetry. Since no motion occurs at the axis of symmetry, a wall could be placed there without altering the solution.

where sgn(x) is the sign function (sign of x) defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0. \end{cases} \tag{5.6.14}$$

"Wave fronts," consisting of discontinuities in both surface elevation and fluid velocity, propagate out from the initial discontinuity as shown. The fluid at any point remains at rest until a wave front passes, after which the surface elevation is zero and there is a current directed toward the region of low surface elevation.

The case shown in Fig. 5.9b has the initial perturbation confined to a finite region, the initial surface elevation being given by

$$\eta = \begin{cases} \eta_0, & |x| < L, \\ 0, & |x| > L. \end{cases} \tag{5.6.15}$$

In this case there is symmetry about the center line, which could therefore be replaced by a solid boundary, with no motion taking place across this line.

5.7 Energetics of Shallow-Water Motion

The energy equations for shallow-water motion can be derived directly from the momentum equations (5.6.4) and (5.6.5) and the continuity equation (5.6.8). The *mechanical* energy equation [cf. (4.6.3)] is obtained by multiplying (5.6.4) by ρHu, (5.6.5) by ρHv, and adding. This gives

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho H(u^2 + v^2)\right] = -\rho g\left[Hu\frac{\partial \eta}{\partial x} + Hv\frac{\partial \eta}{\partial y}\right], \tag{5.7.1}$$

the quantity $\frac{1}{2}\rho H(u^2 + v^2)$ being the *kinetic* energy per unit area. Now by the definition in Section 4.7, the *potential* energy per unit area is

$$\int_{-H}^{\eta} \rho\Phi\, dz = \int_{-H}^{\eta} \rho g z\, dz = \tfrac{1}{2}\rho g(\eta^2 - H^2), \tag{5.7.2}$$

and so the perturbation potential energy per unit area is $\frac{1}{2}\rho g\eta^2$. The equation corresponding to the potential energy equation (4.7.2) is obtained by multiplying (5.6.8) by $\rho g\eta$ to give

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho g\eta^2\right) = -\rho g\left[\eta\frac{\partial}{\partial x}(Hu) + \eta\frac{\partial}{\partial y}(Hv)\right]. \tag{5.7.3}$$

The equation for perturbation *total* energy is obtained by adding (5.7.1) and (5.7.3), namely,

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho H(u^2 + v^2) + \tfrac{1}{2}\rho g\eta^2\right] + \frac{\partial}{\partial x}(\rho g H u\eta) + \frac{\partial}{\partial y}(\rho g H v\eta) = 0. \tag{5.7.4}$$

For the special case in which there is no variation with y, the integral of (5.7.4) with respect to x over the region $|x| < X$ gives

$$\partial E/\partial t + F(X,t) - F(-X,t) = 0, \tag{5.7.5}$$

where

$$E = \int_{-X}^{X} \left[\tfrac{1}{2}\rho H(u^2 + v^2) + \tfrac{1}{2}\rho g\eta^2\right] dx \tag{5.7.6}$$

is the total perturbation energy per unit length in the y direction in the region $|x| < X$ and

$$F(x,t) = \rho g H u\eta \tag{5.7.7}$$

is the rate per unit length in the y direction of transfer of energy in the x direction at point x.

For the case shown in Fig. 5.9a, the perturbation potential energy per unit area is $\frac{1}{2}\rho g\eta_0^2$ in the undisturbed region in which the kinetic energy is zero. After the wave has passed, the perturbation potential energy has dropped to zero, but the kinetic