KOLMOGOROV CASCADE

The velocity fluctuations of a high Reynolds number flow in a three-dimensional velocity field are typically dispersed over all possible wavelengths of the system, from the smallest scales, where viscosity dominates the advection and dissipates the energy of fluid motion, to the effective size of the system. This is not so bizarre: our everyday experience tells us it is so. On the corner of a city street, one might watch the fluttering and whirling of a discarded tram ticket as it is swept by an updraught, driven by localized thermal gradients from traffic or air-conditioning units; later, on the television news, one might see reports or predictions of storms on the city or district scale, and a weather map with isobars spanning whole continents. If you are a sailor you will know how to sail, or not, the multi-scaled surface of a turbulent ocean (Figure 1). The mechanism for this dispersal is vortex stretching and tilting: a conservative process whereby interactions between vorticity and velocity gradients create smaller and smaller eddies with amplified vorticity, until viscosity takes over (Tennekes & Lumley, 1972; Chorin, 1994).

An alternative, crude but picturesque, description of multi-scale turbulence was offered by the early 20th century meteorologist Lewis Fry Richardson (1922) in an evocative piece of doggerel: "big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity". Richardson's often-quoted rhyme is apparently a parody of Irish satirist Jonathan Swift's verse: "So, naturalists observe, a flea—Has smaller fleas that on him prey—And these have smaller still to bite—And so proceed ad infinitum."

The statistics of the velocity fluctuation distribution in turbulent flows were quantified rather more elegantly and rigorously by the mathematician Andrei N. Kolmogorov (1941,b), who derived the subsequently famous "$-5/3$ law" for the energy spectrum of the intermediate scales, or inertial scale subrange, of high Reynolds number flows which are ideally homogeneous (or statistically invariant under translation) and isotropic (or statistically invariant under rotation and reflection) in three velocity dimensions. Two thorough, but different in style and emphasis, accounts of Kolmogorov's turbulence work are Monin & Yaglom (1971) and Frisch (1995).

Kolmogorov's idea was that the velocity fluctuations in the inertial subrange are independent of initial and boundary conditions (i.e., they have no memory of the effects of anisotropic excitation at smaller wave numbers). The turbulent motions in this subrange therefore show universal statistics and the flow is self-similar. From this premise Kolmogorov proposed the first hypothesis of similarity as: "For the locally isotropic turbulence the [velocity fluctuation] distributions F_n are uniquely determined by the quantities ν, the kinematic viscosity, and ε, the rate of average dispersion of energy per unit mass [energy flux]." His second hypothesis of similarity is: "For pulsations [velocity fluctuations] of intermediate orders where the length scale is large compared to the scale of the finest pulsations, whose energy is directly dispersed into heat due to viscosity, the distribution laws F_n are uniquely determined by F and do not depend on ν."

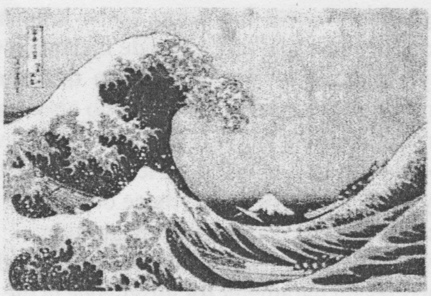

Figure 1. Turbulent action on many different scales in a high Reynolds number flow: woodcut print by Katsushika Hokusai (1760–1849).

Kolmogorov derived the form of the distribution or energy spectrum, which we denote as $\mathcal{E}(k)$, where k is the wave number given by $k^2 = k_x^2 + k_y^2 + k_z^2$, over the inertial subrange simply by dimensional analysis. By the first and second hypotheses, the spectrum must be a function of the energy flux and wave number and independent of the viscosity or any other parameters:

$$\mathcal{E}(k) = f(\varepsilon, k)\,.$$

By reference to the table below (after Vallis, 1999) we find

$$\begin{aligned}\mathcal{E}(k) &\sim \varepsilon^{2/3} g(k) \quad \text{(since } k \text{ is time-independent)}\\ &= C\varepsilon^{2/3} k^{-5/3} \qquad (1)\end{aligned}$$

where C is a dimensionless constant which Kolmogorov (and subsequently many others, see Sreenivasan, 1995) deduced from experimental data to be of order 1.

Quantity	Dimension
Wave number	1/length
Energy per unit mass	length^2/time^2
Energy spectrum $\mathcal{E}(k)$	length^3/time^2
Energy flux ε	energy/time $\sim$ length^2/time^3

The physical picture associated with Equation (1) is that the kinetic energy of large-scale motions (whirls or eddies) is successively subdivided and redistributed among stepwise increasing wave number components (or smaller and smaller whirls and eddies), until the action of viscosity becomes competitive.

Although this process has come to be known as the "Kolmogorov cascade", the cascade metaphor was not used by Kolmogorov. Its first use in this context is apparently due to Onsager (1945), who also highlights another assumption underlying the $-5/3$ law: that the modulation of a given Fourier component of the velocity field is mostly due to those others that belong to wave numbers of comparable magnitude.

So Kolmogorov's energy distribution says that ε is the only relevant parameter for turbulence in the inertial scale range. Can this really be true? Does the notorious "problem of turbulence" really boil down to such a simple relation for intermediate wavenumbers? (The fabled Problem of Turbulence was well summed up by Horace Lamb in 1932: "When I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electro-dynamics and the other is turbulence of fluids. About the former, I am really rather optimistic.") Understandably, for a turbulence result that seems so simple and universal, so flimsily derived yet so powerful, much effort has gone into verifying the wave number spectrum, Equation (1). It is difficult to create extremely high Reynolds number flows in the laboratory, but they exist naturally in the ocean. The first and still the most exciting verification of Equation (1) was carried out by Grant et al. (1962), who made a remarkable series of measurements of turbulent velocities from a ship in the Seymour Narrows, part of the Discovery Passage on the west coast of Canada, where the Reynolds number is $\sim 10^8$ (see Figure 2). A spectral exponent close to $-5/3$ has since been measured many times in materially different flows with high Reynolds number (e.g., Zocchi et al. (1994) in helium).

All this would seem to wrap up the problem of turbulence in the inertial scale range. Or does it? There must surely be a catch somewhere! As usual, the devil is in the details. Kolmogorov himself made a "refinement", as he delicately put it, of his hypotheses (Kolmogorov, 1962). It relates to the problem of small-scale intermittency, or the uneven distribution in space of the small scales. Clearly, intermittency is inherited from initial and boundary conditions, and the side-effects on ε, the assumed-constant rate of energy transfer, are not insignificant.

In fact, there is now quite a log of complaints about the $-5/3$ law, despite its all-pervasive influence on turbulence theoretical and experimental research:

- The hypothesis of local isotropy refers to infinite Reynolds number so is not applicable to a real fluid.
- No-one has ever extracted the $-5/3$ law from the Navier–Stokes equation, or vice versa.
- Is it not a circular argument that to define an inertial subrange one has to assume a cascade process, and to postulate a cascade one has to assume that an inertial subrange exists?
- What about stochastic backscatter?
- Direct interaction between large and small scales can short-circuit the cascade.
- Katul et al. (2003) found that the effects of boundary conditions were evident in the inertial subrange of an atmospheric surface layer.
- The $-5/3$ law is demonstrably invalid in two dimensions. And so on.

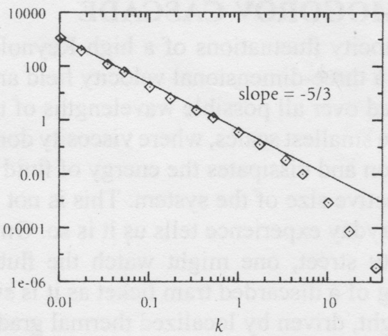

Figure 2. Data re-plotted from Grant et al. (1962), showing a Kolmogorov cascade over nearly three decades. $\phi(k)$ is the measured one-dimensional spectrum function, related to the three-dimensional spectrum function as $\mathcal{E}(k) = k^2\partial^2\phi(k)/\partial k^2 - k\partial\phi(k)/\partial k$.

What is the verdict on the Kolmogorov cascade? Chorin (1994, pp. 55–57) has a bet each way; in the light of experimental verifications of the $-5/3$ law he considers that it may be correct *despite* flaws in the arguments supporting it. The scenario proposed as being entirely consistent with Kolmogorov's theory is that energy can and does slosh back and forth across the spectrum, once the inertial range has been set up, with the energy dissipation ε being the (presumably average) difference between energy flows in both wave number directions.

The Kolmogorov cascade is starting to sound less and less like a waterfall, which *is* one-way, and more and more like an energy exchange network.

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See also **Chaos vs. turbulence; Navier–Stokes Equation; Turbulence**

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